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J.2456 If $m,n\geq 0, m+n=1$ and $x,y,z>0$, then in ΔABC holds:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)h_c^n}\geq\frac{(27)^{\frac{1}{4}}}{2^n}\cdot\left(\sqrt{F}\right)^{1-2n}$$

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We have $ah_a=bh_b=ch_c=2F$.

We will use a theorem by Mehmet Șahin (Turkey) which appeared in $AMM(2015)$:

Let a,b,c be the sides of a triangle. The triangle with sides $\sqrt{a},\sqrt{b},\sqrt{c}$

has the area $F_1=\frac{1}{2}\sqrt{(r_a+r_b+r_c)r}$.

By inequality $r_a+r_b+r_c\geq\sqrt{3}s$ (item 5.29 from $[1]$) we obtain

$$F_1=\frac{1}{2}\sqrt{(r_a+r_b+r_c)r}\geq\frac{1}{2}\sqrt{\sqrt{3}sr}=\frac{1}{2}\left(\sqrt{3}\right)^{\frac{1}{4}}\sqrt{F}\qquad(1)$$

Applying Tsintsifas inequality $\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\geq 2\sqrt{3}F$ (for the triangle with sides $\sqrt{a},\sqrt{b},\sqrt{c}$, that is $\frac{x}{y+z}a+\frac{y}{z+x}b+\frac{z}{x+y}c\geq 2\sqrt{3}F_1$), it follows that

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)h_c^n}=\frac{x\cdot a^{m+n}}{(y+z)a^nh_a^n}+\frac{y\cdot b^{m+n}}{(z+x)b^nh_b^n}+\frac{z\cdot c^{m+n}}{(x+y)c^nh_c^n}=$$

$$=\frac{1}{2^nF^n}\left(\frac{x}{y+z}a+\frac{y}{z+x}b+\frac{z}{x+y}c\right)\geq\frac{2\sqrt{3}F_1}{2^nF^n}\geq\frac{2\sqrt{3}\cdot\frac{1}{2}\left(\sqrt{3}\right)^{\frac{1}{4}}\sqrt{F}}{2^nF^n}=\frac{(27)^{\frac{1}{4}}}{2^n}\cdot\left(\sqrt{F}\right)^{1-2n}.$$

Equality holds if and only if ΔABC is equilateral and $x=y=z$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969