ROMANIAN MATHEMATICAL MAGAZINE

J.2458 If $t,x,y,z>0$, then in ΔABC holds:

$$\frac{t^4+x^4}{(y+z)^2}a^8+\frac{t^4+y^4}{(z+x)^2}b^8+\frac{t^4+z^4}{(x+y)^2}c^8\ge\frac{128t^2}{3}\cdot F^4$$

Proposed by D.M.Bătinețu-Giurgiu, Carmen Vlad – Romania

Solution by Titu Zvonaru-Romania

Applying $AM-GM$ inequality, the known inequality $3(x^2+y^2+z^2)\ge(x+y+z)^2$ and Bergström's inequality, it follows that:

$$\frac{t^4+x^4}{(y+z)^2}a^8+\frac{t^4+y^4}{(z+x)^2}b^8+\frac{t^4+z^4}{(x+y)^2}c^8\ge\frac{2t^2x^2}{(y+z)^2}a^8+\frac{2t^2y^2}{(z+x)^2}b^8+\frac{2t^2z^2}{(x+y)^2}c^8=$$

$$=2t^2\left(\left(\frac{x}{y+z}a^4\right)^2+\left(\frac{y}{z+x}b^4\right)^2+\left(\frac{z}{x+y}c^4\right)^2\right)\ge\frac{2t^2}{3}\left(\frac{xa^4}{y+z}+\frac{yb^4}{z+x}+\frac{zc^4}{x+y}\right)^2=$$

$$=\frac{2t^2}{3}\left(\frac{xa^4}{y+z}+a^4+\frac{yb^4}{z+x}+b^4+\frac{zc^4}{x+y}+c^4-(a^4+b^4+c^4)\right)^2=$$

$$=\frac{2t^2}{3}\left(\frac{(x+y+z)a^4}{y+z}+\frac{(x+y+z)b^4}{z+x}+\frac{(x+y+z)c^4}{x+y}-(a^4+b^4+c^4)\right)^2=$$

$$=\frac{2t^2}{3}\left((x+y+z)\left(\frac{a^4}{y+z}+\frac{b^4}{z+x}+\frac{c^4}{x+y}\right)-(a^4+b^4+c^4)\right)^2\ge$$

$$=\frac{2t^2}{3}\left((x+y+z)\frac{(a^2+b^2+c^2)^2}{y+z+z+x+x+y}-(a^4+b^4+c^4)\right)^2=$$

$$=\frac{2t^2}{3}\left(\frac{(a^2+b^2+c^2)^2}{2}-(a^4+b^2+c^4)\right)=$$

$$=\frac{2t^2}{3}\left(\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{2}\right)^2=\frac{2t^2}{3}\cdot\left(\frac{16F^2}{2}\right)^2=\frac{128t^2}{3}\cdot F^4$$