ROMANIAN MATHEMATICAL MAGAZINE

J.2459 If $m,t,u\geq 0, t+u=4(m+1)$ and $x,y,z>0$, then in ABC holds:

$$\frac{x^{m+1}a^t}{(y+z)^{m+1}\cdot h_a^u}+\frac{y^{m+1}b^t}{(z+x)^{m+1}\cdot h_b^u}+\frac{z^{m+1}c^t}{(x+y)^{m+1}\cdot h_c^y}>\frac{2^{3m-u+3}\left(\sqrt{F}\right)^{t+u}}{3^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniela Barbu – Romania

Solution by Titu Zvonaru-Romania

We have $ah_a=bh_b=ch_c=2F$. By Power Mean inequality we obtain

$$\left(\frac{x^{m+1}+y^{m+1}+z^{m+1}}{3}\right)^{\frac{1}{m+1}}\geq\frac{x+y+z}{3}\Leftrightarrow$$

$$\Leftrightarrow 3^m(x^{m+1}+y^{m+1}+z^{m+1})\geq(x+y+z)^{m+1}\quad(1)$$

Applying inequality (1), Bergström inequality and formula:

$16F^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$, it follows that:

$$\frac{x^{m+1}a^t}{(y+z)^{m+1}\cdot h_a^u}+\frac{y^{m+1}b^t}{(z+x)^{m+1}\cdot h_b^u}+\frac{z^{m+1}c^t}{(x+y)^{m+1}\cdot h_c^u}=$$

$$=\frac{x^{m+1}a^{t+u}}{(y+z)^{m+1}\cdot a^u h_a^u}+\frac{y^{m+1}b^{t+u}}{(z+x)^{m+1}\cdot b^u h_b^u}+\frac{z^{m+1}c^{t+u}}{(x+y)^{m+1}\cdot c^u h_c^u}=$$

$$=\frac{(xa^4)^{m+1}}{(y+z)^{m+1}\cdot 2^uF^u}+\frac{(yb^4)^{m+1}}{(z+x)^{m+1}\cdot 2^uF^u}+\frac{(zc^4)^{m+1}}{(x+y)^{m+1}\cdot 2^uF^u}\geq$$

$$\geq\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{x}{y+z}a^4+\frac{y}{z+x}b^4+\frac{z}{x+y}c^4\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{xa^4}{y+z}+a^4+\frac{yb^4}{z+x}+b^4+\frac{zc^4}{x+y}+c^4-(a^4+b^4+c^4)\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{(x+y+z)a^4}{y+z}+\frac{(x+y+z)b^4}{z+x}+\frac{(x+y+z)c^4}{x+y}-(a^4+b^4+c^4)\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left((x+y+z)\left(\frac{a^4}{y+z}+\frac{b^4}{z+x}+\frac{c^4}{x+y}\right)-(a^4+b^4+c^4)\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left((x+y+z)\frac{(a^2+b^2+c^2)^2}{y+z+z+x+x+y}-(a^4+b^4+c^4)\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{(a^2+b^2+c^2)^2}{2}-(a^4+b^4+c^4)\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{2}\right)^{m+1}=$$

$$=\frac{1}{2^uF^u}\cdot\frac{1}{3^m}\left(\frac{16F^2}{2}\right)^{m+1}=\frac{2^{3m-u+3}\left(\sqrt{F}\right)^{t+u}}{3^m}.$$