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J.2460 In ΔABC the following relationship holds:

$$\frac{6r^2s}{R} \le \sum_{cyc} h_a^2 \cdot tan\frac{A}{2} \le 3rs$$

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Using usual formulas and $AM-GM$ we have

$$h_a^2 \tan\frac{A}{2} = \frac{a^2h_a^2}{a^2}\tan\frac{A}{2} = \frac{4F^2r}{a^2(s-a)} = \frac{4rs(s-a)(s-b)(s-c)}{a^2(s-a)} =$$

$$= \frac{4rs(s-b)(s-c)}{a^2} \le \frac{4rs}{a^2}\left(\frac{s-b+s-c}{2}\right)^2 = rs.$$

It follows that $\Sigma h_a^2 \tan\frac{A}{2} \le 3rs$.

By $ab+bc+ca = s^2+r^2+4Rr, a^2+b^2+c^2 = 2s^2-2r^2-8Rr$ we obtain

$$a^2(s-a)+b^2(s-b)+c^2(s-c) = s(a^2+b^2+c^2)-(a^3+b^3+c^3-3abc)-3abc$$
$$= s(a^2+b^2+c^2)-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)-3abc$$
$$= -s(a^2+b^2+c^2)+2s(ab+bc+ca)-12sRr$$
$$= s(2s^2+2r^2+8Rr-2s^2+2r^2+8Rr-12Rr) = 4sr(R+r).$$

Applying Bergström inequality and Euler inequality it follows that

$$\Sigma h_a^2 \tan\frac{A}{2} = 4F^2r\left(\frac{1}{a^2(s-a)}+\frac{1}{b^2(s-b)}+\frac{1}{c^2(s-c)}\right) \ge$$

$$\ge 4F^2r\frac{9}{a^2(s-a)+b^2(s-b)+c^2(s-c)} =$$

$$= \frac{9r^3s^2}{rs(R+r)} = \frac{9r^2s}{R+r} \ge \frac{9r^2s}{R+\frac{R}{2}} = \frac{6r^2s}{R}$$

Equality holds if and only if the triangle ABC is equilateral.