ROMANIAN MATHEMATICAL MAGAZINE

J.2465 In any ΔABC the following relationship holds:

$$\frac{m_a^2(m_a^2+m_bm_c)}{(m_b+m_c)^2}+\frac{m_b^2(m_b^2+m_cm_a)}{(m_c+m_a)^2}+\frac{m_c^2(m_c^2+m_am_b)}{(m_a+m_b)^2}\ge\frac{27r^2}{2}.$$

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We will prove the inequality

$$\frac{x^2(x^2+yz)}{(y+z)^2}+\frac{y^2(y^2+zx)}{(z+x)^2}+\frac{z^2(z^2+xy)}{(x+z)^2}\ge\frac{x^2+y^2+z^2}{2}\qquad(1)$$

We have

$$\sum_{cyc}\frac{x^2(x^2+yz)}{(y+z)^2}-\sum_{cyc}\frac{x^2}{2}=\sum_{cyc}\frac{x^2(2x^2+2yz-y^2-2yz-z^2)}{2(y+z)^2}$$

$$=\sum_{cyc}\frac{x^2(x^2-y^2+x^2-z^2)}{2(y+z)^2}=$$

$$=\sum_{cyc}\frac{x^2(x^2-y^2)}{2(y+z)^2}+\sum_{cyc}\frac{x^2(x^2-z^2)}{2(y+z)^2}=\sum_{cyc}\frac{x^2(x^2-y^2)}{2(y+z)^2}+\sum_{cyc}\frac{y^2(y^2-x^2)}{2(z+x)^2}=$$

$$=\sum_{cyc}\frac{x^2-y^2}{2}\left(\frac{x^2}{(y+z)^2}-\frac{y^2}{(z+x)^2}\right)$$

$$=\sum_{cyc}\frac{(x^2-y^2)((x^4-y^4)+2z(x^3-y^3)+z^2(x^2-y^2))}{2(y+z)^2(z+x)^2}\ge 0$$

Because the expressions $x^2-y^2, x^4-y^4, x^3-y^3$ have the same sign.

Taking in (1) $x=m_a, y=m_b, z=m_c$ and using the inequalities $3(x^2+y^2+z^2)\ge(x+y+z)^2$ and $m_a+m_b+m_c\ge 9r$ (item 8.3 from $[1]$) it follows that

$$\frac{m_a^2(m_a^2+m_bm_c)}{(m_b+m_c)^2}+\frac{m_b^2(m_b^2+m_cm_a)}{(m_c+m_a)^2}+\frac{m_c^2(m_c^2+m_am_b)}{(m_a+m_b)^2}\ge\frac{m_a^2+m_b^2+m_c^2}{2}\ge$$

$$\ge\frac{(m_a+m_b+m_c)^2}{6}\ge\frac{81r^2}{6}=\frac{27r^2}{2}.$$

Equality holds if and only if ΔABC is equilateral.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969