ROMANIAN MATHEMATICAL MAGAZINE

J.2468 In any ΔABC and $n \in N$ the following relationship holds:

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{3^{n+1}\cdot r^n}{2}$$

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We will prove the inequality

$$\frac{x^n(x^2+yz)}{(y+z)^2}+\frac{y^n(y^2+zx)}{(z+x)^2}+\frac{z^n(z^2+xy)}{(x+z)^2}\geq\frac{x^n+y^n+z^n}{2}\qquad(1)$$

We have

$$\sum_{cyc}\frac{x^n(x^2+yz)}{(y+z)^2}-\sum_{cyc}\frac{x^n}{2}=\sum_{cyc}\frac{x^n(2x^2+2yz-y^2-2yz-z^2)}{2(y+z)^2}=\sum_{cyc}\frac{x^n(x^2-y^2+x^2-z^2)}{2(y+z)^2}=$$

$$=\sum_{cyc}\frac{x^n(x^2-y^2)}{2(y+z)^2}+\sum_{cyc}\frac{x^n(x^2-z^2)}{2(y+z)^2}=\sum_{cyc}\frac{x^n(x^2-y^2)}{2(y+z)^2}+\sum_{cyc}\frac{y^n(y^2-x^2)}{2(z+x)^2}=$$

$$=\sum_{cyc}\frac{x^2-y^2}{2}\left(\frac{x^n}{(y+z)^2}-\frac{y^n}{(z+x)^2}\right)=$$

$$=\sum_{cyc}\frac{(x^2-y^2)((x^{n+2}-y^{n+2})+2z(x^{n+1}-y^{n+1})+z^2(x^n-y^n))}{2(y+z)^2(z+x)^2}\geq 0,$$

because the expressions $x^2-y^2, x^{n+2}-y^{n+2}, x^{n+1}-y^{n+1}, x^n-y^n$ have the same sign.

By Power Mean inequality we obtain

$$\left(\frac{x^n+y^n+z^n}{3}\right)^{1/n}\geq\frac{x+y+z}{3}\Leftrightarrow 3^{n-1}(x^n+y^n+z^n)\geq(x+y+z)^n\qquad(2)$$

Taking $x=w_a, y=w_b, z=w_c$, by $(1),(2)$ and inequality $w_a+w_b+w_c\geq 9r$ (item 8.3 from $[1]$) it follows that

$$\frac{w_a^n(w_a^2+w_bw_c)}{(w_b+w_c)^2}+\frac{w_b^n(w_b^2+w_cw_a)}{(w_c+w_a)^2}+\frac{w_c^n(w_c^2+w_aw_b)}{(w_a+w_b)^2}\geq\frac{w_a^n+w_b^n+w_c^n}{2}\geq$$

$$\geq\frac{(w_a+w_b+w_c)^n}{2\cdot 3^{n-1}}\geq\frac{(9r)^n}{2\cdot 3^{n-1}}=\frac{3^{n+1}\cdot r^n}{2}.$$

Equality holds if and only if ΔABC is equilateral.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969