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J.2472 If $m \geq 0$ and $x, y, z > 0$, then in triangle ABC holds:

$$\frac{x^{m+1}a^4}{(y+z)^{m+1}} + \frac{y^{m+1}b^4}{(z+x)^{m+1}} + \frac{z^{m+1}c^4}{(x+y)^{m+1}} \geq 2^{3-m} \cdot F^2$$

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By Power Mean inequality we obtain

$$\left(\frac{y^{m+1}+z^{m+1}}{2}\right)^{\frac{1}{m+1}} \geq \frac{y+z}{2} \Leftrightarrow 2^m\left(y^{m+1}+z^{m+1}\right) \geq (y+z)^{m+1} \quad (1)$$

Applying (1), Bergström's inequality and formula

$16F^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$, it follows that

$$\frac{x^{m+1}a^4}{(y+z)^{m+1}} + \frac{y^{m+1}b^4}{(z+x)^{m+1}} + \frac{z^{m+1}c^4}{(x+y)^{m+1}} \geq$$

$$\geq \frac{1}{2^m}\left(\frac{x^{m+1}a^4}{y^{m+1}+z^{m+1}} + \frac{y^{m+1}b^4}{z^{m+1}+x^{m+1}} + \frac{z^{m+1}c^4}{x^{m+1}+y^{m+1}}\right) =$$

$$= \frac{1}{2^m}\left(\frac{x^{m+1}a^4}{y^{m+1}+z^{m+1}} + a^4 + \frac{y^{m+1}b^4}{z^{m+1}+x^{m+1}} + b^4 + \frac{z^{m+1}c^4}{x^{m+1}+y^{m+1}} + c^4 - a^4 - b^4 - c^4\right) =$$

$$= \frac{1}{2^m}\left(\left(x^{m+1}+y^{m+1}+z^{m+1}\right)\left(\frac{a^4}{y^{m+1}+z^{m+1}} + \frac{b^4}{z^{m+1}+x^{m+1}} + \frac{c^4}{x^{m+1}+y^{m+1}}\right) - a^4 - b^4 - c^4\right) \geq$$

$$\geq \frac{1}{2^m}\left(\left(x^{m+1}+y^{m+1}+z^{m+1}\right)\left(\frac{(a^2+b^2+c^2)^2}{2\left(x^{m+1}+y^{m+1}+z^{m+1}\right)}\right) - a^4 - b^4 - c^4\right) =$$

$$= \frac{1}{2^m} \cdot \frac{(a^2+b^2+c^2)^2 - 2a^4 - 2b^4 - 2c^4}{2} = \frac{1}{2^m} \cdot \frac{16F^2}{2} = 2^{3-m} \cdot F^2.$$

Equality holds if and only if triangle ABC is equilateral and $x = y = z$.