J.2473 If $x, y, z > 0$, then in ΔABC holds:

$$\left(\frac{ax}{h_a}+\frac{by}{h_b}+\frac{cz}{h_c}\right)^2 \geq 4(xy+yz+zx)$$

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We have $ah_a = bh_b = ch_c = 2F$. Applying Oppenheim's inequality:

$a^2x + b^2y + c^2z \geq 4F\sqrt{xy+yz+zx}$, it follows that:

$$\left(\frac{ax}{h_a}+\frac{by}{h_b}+\frac{cz}{h_c}\right)^2 = \left(\frac{a^2x}{ah_a}+\frac{b^2y}{bh_b}+\frac{c^2z}{ch_c}\right)^2 =$$

$$= \frac{1}{4F^2}(a^2x+b^2y+c^2z)^2 \geq \frac{1}{4F^2}\left(4F\sqrt{xy+yz+zx}\right)^2 = 4(xy+yz+zx).$$

Equality holds if and only if ΔABC is equilateral and $x = y = z$.