ROMANIAN MATHEMATICAL MAGAZINE

J.2478 Solve for real numbers:

$$\sqrt{x-y}+3\sqrt{y-z}+5\sqrt{z+x}=x+\frac{35}{2}$$

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We have $x \geq y \geq z$. Applying Cauchy-Buniakovski-Schwarz inequality, we obtain:

$$(1+9+25)\big((x-y)+(y-z)+(z+x)\big) \geq \left(\sqrt{x-y}+3\sqrt{y-z}+5\sqrt{z+x}\right)^2 \quad (1)$$

It follows that

$$\left(x+\frac{35}{2}\right)^2=\left(\sqrt{x-y}+3\sqrt{y-z}+5\sqrt{z+x}\right)^2 \leq 35(x-y+y-z+z+x)=70x,$$

that is

$$x^2+35x+\left(\frac{35}{2}\right)^2 \leq 70x \Leftrightarrow \left(x-\frac{35}{2}\right)^2 \leq 0,$$

hence $x=\frac{35}{2}$. Yields that in (1) there is equality. It follows that

$$\frac{x-y}{1}=\frac{y-z}{9}=\frac{z+x}{25}=\frac{x-y+y-z+z+x}{1+9+25}=\frac{2x}{35}=1.$$

It results that $x=\frac{35}{2}, y=\frac{33}{2}, z=\frac{15}{2}$.