ROMANIAN MATHEMATICAL MAGAZINE

J.2483 Let ABC be a triangle. Find the values of a, b, c such that:

$$\frac{m_a}{h_a} \in N$$

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Suppose that $\frac{m_a}{h_a}$ is an positive integer; then $\frac{m_a^2}{h_a^2}$ is an integer.

Using Heron formula $16F^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$, we obtain

$$\frac{m_a^2}{h_a^2} = \frac{4m_a^2}{4h_a^2} = \frac{2b^2 + 2c^2 - a^2}{\frac{16F^2}{a^2}} = \frac{2a^2b^2 + 2a^2c^2 - a^4}{16F^2} =$$

$$= \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 + b^4 + c^4 - 2b^2c^2}{16F^2} = 1 + \frac{(b^2 - c^2)^2}{16F^2}.$$

Since $\frac{m_a^2}{h_a^2}$ is an integer, yields that $\frac{(b^2-c^2)^2}{16F^2}$ is an integer and

$$\frac{m_a^2}{h_a^2} - \frac{(b^2 - c^2)^2}{16F^2} = 1 \quad (1)$$

Because if two perfect squares have the difference equal to 1 if and only if one of them is equal to 0. It follows that $b = c$, and $\frac{m_a}{h_a} \in N$ if and only if the triangle ABC is isosceles with $b = c$.