ROMANIAN MATHEMATICAL MAGAZINE

J.2485 If $x,y,z>0, xyz=x+y+z+2$ then $xy+yz+zx\geq 12.$

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It is easy to check that the relationship $xyz=x+y+z+2$ is equivalent to

$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$. Indeed, we have

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1\Leftrightarrow$$

$$(x+1)(y+1)+(y+1)(z+1)+(z+1)(x+1)=(x+1)(y+1)(z+1)\Leftrightarrow$$

$$xy+x+y+1+yz+y+z+1+zx+z+x+1=xyz+xy+yz+zx+x+y+z+1$$

$$\Leftrightarrow xyz=x+y+z+2.$$

We denote $a=\frac{1}{x+1}, b=\frac{1}{y+1}, c=\frac{1}{z+1}$; then $a+b+c=1$ and

$$x+1=\frac{1}{a}\Rightarrow x=\frac{1}{a}-1=\frac{1-a}{a}=\frac{b+c}{a}.$$

It results that if $xyz=x+y+z+2$ then there exist $a,b,c>0$ such that

$$x=\frac{b+c}{a}, y=\frac{c+a}{b}, z=\frac{a+b}{c}.$$

Using $AM-GM$ inequality, it follows that

$$xy+yz+zx=\frac{(b+c)(c+a)}{ab}+\frac{(c+a)(a+b)}{bc}+\frac{(a+b)(b+c)}{ca}=$$

$$=\frac{c(b+c)(c+a)+a(c+a)(a+b)+b(z+b)(b+c)}{abc}=$$

$$=\frac{a^3+b^3+c^3+(a+b+c)(ab+bc+ca)}{abc}\geq\frac{3abc+9abc}{abc}=12.$$

Equality holds if and only if $a=b=c$, that is $x=y=z=2$.