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J.2492 If $a,b,c>0, abc=1$ then

$$\left(a^5+1\right)\left(b^5+1\right)\left(c^5+1\right)\ge\left(a^3+\frac{1}{a}\right)\left(b^3+\frac{1}{b}\right)\left(c^3+\frac{1}{c}\right)$$

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We will prove a more general inequality:

If $a,b,c>0, abc=1$ and $m\in N$ then

$$(a^{m+2}+1)(b^{m+2}+1)(c^{m+2}+1)\ge\left(a^m+\frac{1}{a}\right)\left(b^m+\frac{1}{b}\right)\left(c^m+\frac{1}{c}\right)\quad(1)$$

Applying Holder inequality it follows that

$$\underbrace{(a^{m+2}+1)(a^{m+2}+1)\dots(a^{m+2}+1)}_{m}(1+b^{m+2})(1+c^{m+2})$$
$$\ge\left(\left(a^{m(m+2)}\right)^{\frac{1}{m+2}}+\left(b^{m+2}c^{m+2}\right)^{\frac{1}{m+2}}\right)^{m+2}=(a^m+bc)^{m+2}\quad(2)$$

and similarly

$$\underbrace{(b^{m+2}+1)(b^{m+2}+1)\dots(b^{m+2}+1)}_{m}(1+c^{m+2})(1+a^{m+2})\ge(b^m+ca)^{m+2}\quad(3)$$

$$\underbrace{(c^{m+2}+1)(c^{m+2}+1)\dots(c^{m+2}+1)}_{m}(1+a^{m+2})(1+b^{m+2})\ge(c^m+ab)^{m+2}\quad(4)$$

Since $abc=1$, multiplying $(2),(3),(4)$, we obtain the inequality (1).

For $m=3$ it results the given inequality.

Equality holds if and only if $a=b=c=1$.