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J.2511 Let $ABCD$ be a rectangle with $AB = l, BC = 2l$ and M an interior point such that $< MAB = < MBA = 15°$. Find the angles' measures of triangle MCD.

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Since $< MAB = < MBA = 15°$, the point M belongs to the perpendicular bisector of AB; te triangle MCD is isosceles with $MC = MD$.

Solution 1.

Let P, Q be the midpoints of sides BC and AD, respectively. We consider the point M' in interior of the rectangle, such that the triangle $PM'Q$ is equilateral. Yields that triangle QAM' is isosceles with $< QAM' = 30°$, hence $< QAM' = 75°$. It follows that $<M'AB = 15°$ and point M' coincide with the point M. It results that in triangle AMD, the median MQ is equal to half of AD. We deduce that triangle AMD is right-angled, $< AMD = 90°, < ADM = 15°$, that is $< MDC = 75°$.

Solution 2.

We denote $x =< ADM$. Since $AM = \frac{l}{2\cos 15°} = \frac{l\sin 15°}{2\sin 15°\cos 15°} = \frac{l\sin 15°}{\sin 30°} = 2l\sin 15°$, by the sines law in triangle AMD we obtain

$$\frac{AM}{\sin x} = \frac{AD}{\sin(x + 75°)} \Leftrightarrow \frac{2l\sin 15°}{\sin x} = \frac{2l}{\sin(x + 75°)} \Leftrightarrow \sin 15° \sin(x + 75°) = \sin x$$

$$\cos(x + 60°) - \cos(x + 90°) = 2\sin x \Leftrightarrow \cos(x + 60°) + \sin x = 2\sin x \Leftrightarrow$$

$$\sin(30° - x) = \sin x \Leftrightarrow x = 15°,$$

hence $< MDC = 75°$.

Solution 3.

Since $AM = \frac{l}{2\cos 15°}$, by cosines law we obtain

$$MD^2 = AM^2 + AD^2 - 2AM \cdot AD\cos 75° = \frac{l^2}{4\cos^2 15°} + 4l^2 - \frac{4l^2\cos 75°}{2\cos 15°} =$$

$$= \frac{l^2}{4\cos^2 15°} + 4l^2 - \frac{4l^2\sin 15°\cos 15°}{2\cos^2 15°} = \frac{l^2}{4\cos^2 15°} + 4l^2 - \frac{l^2\sin 30°}{\cos^2 15°} =$$

$$= \frac{l^2}{4\cos^2 15°} + 4l^2 - \frac{l^2}{2\cos^2 15°} = 4l^2 - \frac{l^2}{4\cos^2 15°}.$$

It easy to see that $AM^2 + MD^2 = 4l^2 = AD^2$, that is the triangle AMD is right-angled at D. It follows that $< ADM = 90° - 75° = 15°$, hence $< MDC = 75°$.

Solution 4.

Since $AM=\frac{l}{2\cos 15^\circ}$, by cosines law we obtain

$$MQ^2=AM^2+AQ^2-2AM\cdot AQ\cos 75^\circ=\frac{l^2}{4\cos^2 15^\circ}+l^2-\frac{2l^2\cos 75^\circ}{2\cos 15^\circ}=$$

$$=\frac{l^2}{4\cos^2 15^\circ}+l^2-\frac{2l^2\sin 15^\circ\cos 15^\circ}{2\cos^2 15^\circ}=\frac{l^2}{4\cos^2 15^\circ}+l^2-\frac{l^2\sin 30^\circ}{2\cos^2 15^\circ}=$$

$$=\frac{l^2}{4\cos^2 15^\circ}+l^2-\frac{l^2}{4\cos^2 15^\circ}=l^2=AQ^2.$$

In triangle AMD, the median MQ is equal to half of AD; then the triangle AMD is right-angled, $<MDA$=15° and $<MDC=75^\circ$.