ROMANIAN MATHEMATICAL MAGAZINE

J.2520 Find the natural numbers n such that $n^2+7n+40$ to be written as a product of four natural consecutive numbers.

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Let p a natural number such that

$$n^2+7n+40=p(p+1)(p+2)(p+3) \quad (1)$$

Since $p(p+1)(p+2)(p+3)=(p^2+3p)(p^2+3p+2)$, we denote $m=p^2+3p$. The equation (1) is equivalent to

$$n^2+7n+40=m(m+2)$$

$$4n^2+28n+160=4m^2+8m$$

$$4n^2+28n+49+111=4m^2+8m+4-4$$

$$(2n+7)^2+115=(2m+2)^2$$

$$(2m+2-2n-7)(2m+2+2n+7)=115.$$

Since $115=5\cdot 23$ and $2m+2>2n+7$, we have the possibilities:

$$2m+2-2n-7=1, 2m+2+2n+7=115 \quad (2)$$

$$2m+2-2n-7=5, 2m+2+2n+7=23 \quad (3)$$

From (2) it results $2n+7=57, 2m+2=58$; then $n=25, m=28$. Yields $p^2+3p=28$, hence $p=4$.

From (3) it follows $2n+7=9, 2m+2=14$; then $n=1, m=6$. Yields $p^2+3p=6$, and there is not integer solutions.

It results that the equation (1) has the solution $(n,p)=(25,4)$.