ROMANIAN MATHEMATICAL MAGAZINE

J.2539 If $x, y, z > 0$, then:

$$2\sum_{cyc}(x+y)^4 \geq 32xyz(x+y+z) + \sum_{cyc}(y-x)(x+y+2z).$$

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We have:

$$\sum_{cyc}(y-x)(x+y+2z)$$
$$= y^2 - x^2 + 2z(y-x) + z^2 - y^2 + 2x(z-y) + x^2 - z^2 + 2y(x-z) = 0.$$

It remains to prove that:

$$\sum_{cyc}(x+y)^4 \geq 16xyz(x+y+z) \quad (1)$$

Three proofs for (1):

a) Since (1) is symmetric and homogenous of 4 degree, it suffices to prove the inequality for

$z = 0$ and for $y = x$. For $z = 0$, the inequality (1) is true. For $y = x$ we have to prove that

$$2(x+y)^4 + 16y^4 \geq 16xy^2(x+2y) \Leftrightarrow (x+y)^4 + 8y^4 \geq 8xy^2(x+2y)$$

$$(x-y)^2(x+3y)^2 \geq 0.$$

Equality holds if and only if $x = y = z$.

b) Using the known inequality $3(a^2+b^2+c^2) \geq (a+b+c)^2$, it suffices to prove that

$$((a+b)^2 + (b+c)^2 + (c+a)^2)^2 \geq 48xyz(x+y+z)$$

$$(a^2+b^2+c^2+ab+bc+ca)^2 \geq 12xyz(x+y+z) \quad (2)$$

Applying the known inequalities $a^2+b^2+c^2 \geq ab+bc+ca$ and

$(ab+bc+ca)^2 \geq 3abc(a+b+c)$, it follows that

$$(a^2+b^2+c^2+ab+bc+ca)^2 \geq 4(ab+bc+ca)^2 \geq 12xyz(x+y+z),$$

the inequality (2) is true.

c) The inequality (1) is equivalent to

$$2\sum_{\text{cyc}} x^4 + 4\sum_{\text{sym}} x^3y + 6\sum_{\text{cyc}} x^2y^2 \geq 16\sum_{\text{cyc}} x^2yz$$

$$\sum_{\text{sym}} x^4 + 4\sum_{\text{sym}} x^3y + 3\sum_{\text{sym}} x^2y^2 \geq 8\sum_{\text{sym}} x^2yz \qquad (3)$$

The inequality (3) results by Muirhead's inequality:

$$\sum_{\text{sym}} x^4 \geq \sum_{\text{sym}} x^2yz,\ 4\sum_{\text{sym}} x^3y \geq 4\sum_{\text{sym}} x^2yz,\ 3\sum_{\text{sym}} x^2y^2 \geq 3\sum_{\text{sym}} x^2yz.$$