ROMANIAN MATHEMATICAL MAGAZINE

J.2564 In triangle ABC the following relationship holds:

$$\frac{32s^5 - a^5 - b^5 - c^5}{8s^3 - a^3 - b^3 - c^3} \geq 120r^2$$

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Let $P(a,b,c) = (a+b+c)^5 - a^5 - b^5 - c^5$. Since

$$P(a,b,c) = (a+b+c)^5 - a^5 - b^5 - c^5 =$$

$$= (a+b+c-a)((a+b+c)^4 + (a+b+c)^3c + (a+b+c)^2c^2 + (a+b+c)c^3 + c^4) + (b+c)(b^4 + b^3c + b^2c^2 + bc^3 + c^4) =$$

$$= (b+c)((a+b+c)^4 + (a+b+c)^3c + (a+b+c)^2c^2 + (a+b+c)c^3 + c^4 + b^4 + b^3c + b^2c^2 + bc^3 + c^4),$$

we deduce that $P(a,b,c)$ is divisible by $b+c$.

Similarly, $P(a,b,c)$ s divisible by $c+a$ and $a+b$.

We want to find m,n such that

$$(a+b+c)^5 - a^5 - b^5 - c^5 =$$

$$= (a+b)(b+c)(c+a)\left(m(a^2+b^2+c^2) + n(ab+bc+ca)\right).$$

For $a=b=c=1$ yields $243 - 3 = 8(3m+3n) \Leftrightarrow m+n=10$,

and for $a=b=1, c=0$ we get $32 - 2 = 2(2m+n) \Leftrightarrow 2m+n=15$.

We obtain $m=n=5$ and

$$(a+b+c)^5 - a^5 - b^5 - c^5 =$$

$$= 5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca) \quad (1)$$

We also obtain

$$(a+b+c)^3 - a^3 - b^3 - c^3 =$$

$$= (b+c)((a+b+c)^2 + (a+b+c)a + a^2 - b^2 + bc - c^2) =$$

$$= (b+c)(a^2+b^2+c^2+2ab+2bc+2ca+a^2+ab+ac+a^2-b^2+bc-c^2) =$$

$$= (b+c)(3a^2+3ab+3bc+3ca) = 3(a+b)(b+c)(c+a) \quad (2)$$

By (1) and (2), the given inequality is equivalent to

$$\frac{(a+b+c)^5-a^5-b^5-c^5}{(a+b+c)^3-a^3-b^3-c^3}\geq 120r^2$$

$$\frac{5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca)}{3(a+b)(b+c)(c+a)}\geq 120r^2$$

$$a^2+b^2+c^2+ab+bc+ca\geq 72r^2 \qquad (3)$$

Since $a^2+b^2+c^2\geq 36r^2$ (item 5.13 from $[1]$) and

$ab+bc+ca\geq 36r^2$ (item 5.16 from $[1]$),

it follows that the inequality (3) is true.

Equality holds if and only if the triangle ABC is equilateral.

$[1]$ **O. Bottema, *Geometric Inequalities*, Groningen 1969**