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J.2566 In any triangle ABC the following relationship holds:

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{c+a}\geq 8\sqrt{3}\cdot F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți – Romania

Solution by Titu Zvonaru-Romania

We have $ah_a=bh_b=ch_c=2F$.

Applying Bergström's inequality, the well known inequality $a^2+b^2+c^2\geq ab+bc+ca$ and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\geq 4\sqrt{3}F$, it follows that

$$\frac{(a^4+b^4)h_c}{a+b}+\frac{(b^4+c^4)h_a}{b+c}+\frac{(c^4+a^4)h_b}{c+a}=$$

$$=\frac{(a^4+b^4)ch_c}{ca+bc}+\frac{(b^4+c^4)ah_a}{ab+ca}+\frac{(c^4+a^4)bh_b}{bc+ab}=$$

$$=2F\left(\frac{a^4}{ca+bc}+\frac{b^4}{ab+ca}+\frac{c^4}{bc+ab}+\frac{b^4}{ca+bc}+\frac{c^4}{ab+ca}+\frac{a^4}{bc+ab}\right)\geq$$

$$\geq 2F\left(\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}+\frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}\right)\geq$$

$$\geq 2F\left(\frac{a^2+b^2+c^2}{2}+\frac{a^2+b^2+c^2}{2}\right)\geq 8\sqrt{3}\cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral.