ROMANIAN MATHEMATICAL MAGAZINE

**J.2573 If $m \geq 0$ then in any triangle $ABC$ holds:**

$$\frac{a^{2m+1}}{h_a(b+c)^m(2a+b+c)^m}+\frac{b^{2m+1}}{h_b(c+a)^m(a+2b+c)^m}+\frac{c^{2m+1}}{h_c(a+b)^m(a+b+2c)^m}\geq\frac{\sqrt{3}}{2^{3m-1}}$$

*Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți – Romania*

*Solution by Titu Zvonaru-Romania*

We have $ah_a=bh_b=ch_c=2F$. Applying Radon's inequality, it follows that

$$\frac{a^{2m+1}}{h_a(b+c)^m(2a+b+c)^m}+\frac{b^{2m+1}}{h_b(c+a)^m(a+2b+c)^m}+\frac{c^{2m+1}}{h_c(a+b)^m(a+b+2c)^m}=$$

$$=\frac{a^{2m+2}}{ah_a(b+c)^m(2a+b+c)^m}+\frac{b^{2m+2}}{bh_b(c+a)^m(a+2b+c)^m}+$$

$$+\frac{c^{2m+2}}{ch_c(a+b)^m(a+b+2c)^m}=$$

$$=\frac{1}{2F}\left(\frac{(a^2)^{m+1}}{(b+c)^m(2a+b+c)^m}+\frac{(b^2)^{m+1}}{(c+a)^m(a+2b+c)^m}+\frac{(c^2)^{m+1}}{(a+b)^m(a+b+2c)^m}\right)=$$

$$\geq\frac{1}{2F}\cdot\frac{(a^2+b^2+c^2)^{m+1}}{\big((b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)\big)^m}\quad(1)$$

Using the known inequality $ab+bc+ca\leq a^2+b^2+c^2$, we obtain

$$(b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)=2(a^2+b^2+c^2+3ab+3bc+3ca)\leq 8(a^2+b^2+c^2).$$

By Ionescu-Weitzenbock's inequality and $(1)$, it results that

$$\frac{a^{2m+1}}{h_a(b+c)^m(2a+b+c)^m}+\frac{b^{2m+1}}{h_b(c+a)^m(a+2b+c)^m}+\frac{c^{2m+1}}{h_c(a+b)^m(a+b+2c)^m}\geq$$

$$\geq\frac{1}{2F}\cdot\frac{a^2+b^2+c^2}{2^{3m}}\geq\frac{1}{2F}\cdot\frac{4\sqrt{3}F}{2^{3m}}=\frac{\sqrt{3}}{2^{3m-1}}.$$

Equality holds if and only if the triangle $ABC$ is equilateral.