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J.2574 If $m,n\geq 0, m+n=2$ and $x,y,z>0$ then in any triangle ABC holds:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)h_c^n}\geq\frac{\sqrt{3}}{2^{n-1}}\cdot F^{1-n}$$

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We have $ah_a=bh_b=ch_c=2F$. Applying Tsintsifas' inequality

$\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}\geq 2\sqrt{3}F$, it follows that:

$$\frac{x\cdot a^m}{(y+z)h_a^n}+\frac{y\cdot b^m}{(z+x)h_b^n}+\frac{z\cdot c^m}{(x+y)h_c^n}=\frac{x\cdot a^{m+n}}{(y+z)a^n h_a^n}+\frac{y\cdot b^{m+n}}{(z+x)b^n h_b^n}+\frac{z\cdot c^{m+n}}{(x+y)c^n h_c^n}=$$

$$=\frac{1}{(2F)^n}\left(\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\right)\geq\frac{1}{(2F)^n}\cdot\left(2\sqrt{3}F\right)=\frac{\sqrt{3}}{2^{n-1}}\cdot F^{1-n}.$$

Equality holds if and only if the triangle ABC is equilateral and $x=y=z$.