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J.2578 If $m \geq 0$ then in any triangle ABC holds:

$$\frac{a^{m+1}}{h_a^{m+1}(b+c)^m(2a+b+c)^m}+\frac{b^{m+1}}{h_b^{m+1}(c+a)^m(a+2b+c)^m}+\frac{c^{m+1}}{h_c^{m+1}(a+b)^m(a+b+2c)^m}\geq\frac{\sqrt{3}}{2^{4m-1}F^m}$$

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We have $ah_a = bh_b = ch_c = 2F$. Applying Radon's inequality, it follows that

$$\frac{a^{m+1}}{h_a^{m+1}(b+c)^m(2a+b+c)^m}+\frac{b^{m+1}}{h_b^{m+1}(c+a)^m(a+2b+c)^m}+$$

$$+\frac{c^{m+1}}{h_c^{m+1}(a+b)^m(a+b+2c)^m}=$$

$$\frac{(a^2)^{m+1}}{a^{m+1}h_a^{m+1}(b+c)^m(2a+b+c)^m}+\frac{(b^2)^{m+1}}{b^{m+1}h_b^{m+1}(c+a)^m(a+2b+c)^m}+$$

$$+\frac{(c^2)^{m+1}}{c^{m+1}h_c^{m+1}(a+b)^m(a+b+2c)^m}\geq$$

$$\geq\frac{1}{(2F)^{m+1}}\cdot\frac{\left(a^2+b^2+c^2\right)^{m+1}}{\big((b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)\big)^m}\quad(1)$$

Using the known inequality $ab+bc+ca \leq a^2+b^2+c^2$, we obtain

$$(b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)=2(a^2+b^2+c^2+$$
$$+3ab+3bc+3ca)\leq 8(a^2+b^2+c^2).$$

By Ionescu-Weitzenbock's inequality and (1), it results that

$$\frac{a^{m+1}}{h_a^{m+1}(b+c)^m(2a+b+c)^m}+\frac{b^{m+1}}{h_b^{m+1}(c+a)^m(a+2b+c)^m}+$$

$$+\frac{c^{m+1}}{h_c^{m+1}(a+b)^m(a+b+2c)^m}\geq$$

$$\geq\frac{1}{(2F)^{m+1}}\cdot\frac{a^2+b^2+c^2}{2^{3m}}\geq\frac{1}{(2F)^{m+1}}\cdot\frac{4\sqrt{3}F}{2^{3m}}=\frac{\sqrt{3}}{2^{4m-1}F^m}$$

Equality holds if and only if the triangle ABC is equilateral.