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$$\frac{a^{2m}}{h_a^2(b+c)^m(2a+b+c)^m}+\frac{b^{2m}}{h_b^2(c+a)^m(a+2b+c)^m}+\frac{c^{2m}}{h_c^2(a+b)^m(a+b+2c)^m}\ge\frac{\sqrt{3}}{2^{3m}F}$$

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We have $ah_a=bh_b=ch_c=2F$. Applying Radon's inequality, it follows that

$$\frac{a^{2m}}{h_a^2(b+c)^m(2a+b+c)^m}+\frac{b^{2m}}{h_b^2(c+a)^m(a+2b+c)^m}+\frac{c^{2m}}{h_c^2(a+b)^m(a+b+2c)^m}=$$

$$=\frac{1}{4F^2}\left(\frac{(a^2)^{m+1}}{(b+c)^m(2a+b+c)^m}+\frac{(b^2)^{m+1}}{(c+a)^m(a+2b+c)^m}+\frac{(c^2)^{m+1}}{(a+b)^m(a+b+2c)^m}\right)\ge$$

$$\ge\frac{1}{4F^2}\cdot\frac{(a^2+b^2+c^2)^{m+1}}{\big((b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)\big)^m}\quad(1)$$

Using the known inequality $ab+bc+ca\le a^2+b^2+c^2$, we obtain

$$(b+c)(2a+b+c)+(c+a)(a+2b+c)+(a+b)(a+b+2c)=2(a^2+b^2+c^2+3ab+3bc+3ca)\le 8(a^2+b^2+c^2).$$

By Ionescu-Weitzenbock's inequality and (1), it results that

$$\frac{a^{2m}}{h_a^2(b+c)^m(2a+b+c)^m}+\frac{b^{2m}}{h_b^2(c+a)^m(a+2b+c)^m}+\frac{c^{2m}}{h_c^2(a+b)^m(a+b+2c)^m}\ge$$

$$\ge\frac{1}{4F^2}\cdot\frac{a^2+b^2+c^2}{2^{3m}}\ge\frac{1}{4F^2}\cdot\frac{4\sqrt{3}F}{2^{3m}}=\frac{\sqrt{3}}{2^{3m}F}.$$

Equality holds if and only if the triangle ABC is equilateral.