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J.2590 In triangle ABC the following relationship holds

$$\left((a+b)(b+c)(c+a)\right)^{\frac{1}{3}} \geq 4\sqrt{3}r$$

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Solution 1: Using Cesaro's inequality $(a+b)(b+c)(c+a) \geq 8abc$, Euler's inequality $R \geq 2r$ and Mitrinovici inequality $s \geq 3\sqrt{3}r$, it follows that

$$(a+b)(b+c)(c+a) \geq 8abc = 32Rrs \geq 64r^2s \geq 192\sqrt{3}r^3,$$

hence

$$\left((a+b)(b+c)(c+a)\right)^{\frac{1}{3}} \geq 4\sqrt{3}r.$$

Solution 2: Using Gerretsen's inequality $s^2 > 16Rr - 5r^2$, Euler's inequality $R \geq 2r$ and Mitrinovici inequality $s \geq 3\sqrt{3}r$ we obtain

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc\\ &= 2s(s^2+r^2+4Rr) - 4Rrs = 2s(s^2+r^2+2Rr)\\ &\geq 6\sqrt{3}r(16Rr - 5r^2 + r^2 + 2Rr) = 6\sqrt{3}r(18Rr - 4r^2) \geq\\ &\geq 6\sqrt{3}r(36r^2 - 4r^2) = 192\sqrt{3}r^3,\end{aligned}$$

Hence

$$\left((a+b)(b+c)(c+a)\right)^{\frac{1}{3}} \geq 4\sqrt{3}r.$$

Equality holds if and only if the triangle ABC is equilateral.