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J.2601 If $x, y, z > 0$ then in triangle ABC holds:

$$\frac{1}{w_a^2}\left(\frac{y+z}{x}\right)+\frac{1}{w_b^2}\left(\frac{z+x}{y}\right)+\frac{1}{w_c^2}\left(\frac{x+y}{z}\right)\geq\frac{18}{s^2}$$

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Applying $AM-GM$ inequality, Cesaro's inequality $(x+y)(y+z)(z+x)\geq 8xyz$, the inequality $w_aw_bw_c\leq rs^2$ (item 8.14 from $[1]$) and the inequality $s^2\geq 27r^2$ (item 5.11 from $[1]$),it follows that

$$\left(\frac{1}{w_a^2}\left(\frac{y+z}{x}\right)+\frac{1}{w_b^2}\left(\frac{z+x}{y}\right)+\frac{1}{w_c^2}\left(\frac{x+y}{z}\right)\right)^3\geq$$

$$\geq\frac{27(x+y)(y+z)(z+x)}{xyzw_a^2w_b^2w_c^2}\geq\frac{27\cdot 8}{r^2s^4}\geq\frac{27\cdot 8}{s^4\cdot\frac{s^2}{27}}=\frac{3^6\cdot 2^3}{s^6},$$

hence

$$\frac{1}{w_a^2}\left(\frac{y+z}{x}\right)+\frac{1}{w_b^2}\left(\frac{z+x}{y}\right)+\frac{1}{w_c^2}\left(\frac{x+y}{z}\right)\geq\frac{18}{s^2}.$$

Equality holds if and only in the triangle ABC is equilateral and $x=y=z$.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969