ROMANIAN MATHEMATICAL MAGAZINE

J.2602 If $x,y>0$ then

$$\left(\frac{1}{2}+\frac{x}{2y}\right)^2\left(\frac{1}{2}+\frac{y}{2x}\right)^2\ge\frac{1}{16}\left(1+\frac{2x+y}{(x^2y)^{\frac{1}{3}}}\right)\left(1+\frac{2y+x}{(xy^2)^{\frac{1}{3}}}\right)\ge 1$$

Proposed by Neculai Stanciu – Romania

Solution by Titu Zvonaru-Romania

It easy to see that the inequalities are equivalent to:

$$\frac{(x+y)^4}{x^2y^2}\ge\left(1+\frac{2x+y}{(x^2y)^{\frac{1}{3}}}\right)\left(1+\frac{2y+x}{(xy^2)^{\frac{1}{3}}}\right)\ge 16\quad(1)$$

By $AM-GM$ inequality we obtain

$$2x+y=x+x+y\ge 3(x^2y)^{\frac{1}{3}}\quad(2)\ \text{and}\ 2y+x=y+y+x\ge 3(xy^2)^{\frac{1}{3}}\ \ (3).$$

It follows that

$$\left(1+\frac{2x+y}{(x^2y)^{\frac{1}{3}}}\right)\left(1+\frac{2y+x}{(xy^2)^{\frac{1}{3}}}\right)\ge\left(1+\frac{3(x^2y)^{\frac{1}{3}}}{(x^2y)^{\frac{1}{3}}}\right)\left(1+\frac{3(xy^2)^{\frac{1}{3}}}{(xy^2)^{\frac{1}{3}}}\right)=16.$$

Using again the inequalities (2) and (3), it follows that

$$\left(1+\frac{2x+y}{(x^2y)^{\frac{1}{3}}}\right)\left(1+\frac{2y+x}{(xy^2)^{\frac{1}{3}}}\right)=\frac{((x^2y)^{\frac{1}{3}}+2x+y)((xy^2)^{\frac{1}{3}}+2y+x)}{xy}\le$$

$$=\frac{\left(\frac{2x+y}{3}+2x+y\right)\left(\frac{2y+x}{3}+2y+x\right)}{xy}=\frac{16(2x+y)(2y+x)}{9xy}\quad(4)$$

By (4), it results that for the left inequality from (1), it suffices to prove that

$$\frac{(x+y)^4}{x^2y^2}\ge\frac{16(2x+y)(2y+x)}{9xy},$$

which is equivalent to

$$9x^4+4x^3y-26x^2y^2+4xy^3+9y^4\ge 0$$

$$(x-y)^2(9x^2+22xy+9y^2)\ge 0,\ \text{true for } x,y>0.$$

Equality holds if and only if $x=y$.