ROMANIAN MATHEMATICAL MAGAZINE

J.2610 If $a,b,c\geq 0$ then prove that:

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\geq\sqrt{2}(a+b+c)$$

Proposed by Neculai Stanciu – Romania

Solutions by Titu Zvonaru-Romania

Solution 1. Applying Cauchy-Buniakovski-Schwarz inequality, it follows that

$$\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2=$$

$$=2(a^2+b^2+c^2)+2\sqrt{(a^2+b^2)(b^2+c^2)}+2\sqrt{(b^2+c^2)(c^2+a^2)}+2\sqrt{(c^2+a^2)(a^2+b^2)}\geq$$

$$\geq 2(a^2+b^2+c^2)+2\sqrt{(ab+bc)^2}+2\sqrt{(bc+ca)^2}+2\sqrt{(ca+ab)^2}=2(a+b+c)^2.$$

Equality holds if and only if $a=b=c$.

Solution 2. Using complex modulus number and triangle inequality, we obtain

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}=|a+ib|+|b+ic|+|c+ia|$$
$$\geq|a+b+c+i(a+b+c)|=$$

$$=\sqrt{(a+b+c)^2+(a+b+c)^2}=\sqrt{2}(a+b+c).$$

Solution 3. Applying Minkovski inequality, it follows that

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\geq\sqrt{(a+b+c)^2+(b+\ c+a)^2}=$$

$$=\sqrt{2(a+b+c)^2}=\sqrt{2}(a+b+c).$$

Equality holds if and only if $a=b=c$.