ROMANIAN MATHEMATICAL MAGAZINE

J.2615 If $x, y, z > 0$ and ABC is a triangle with area F, then:

$$\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2 a^4+2\right)\cdot\left(\left(\frac{y}{z+x}+\frac{y+z}{x}\right)^2+2\right)\cdot\left(\left(\frac{z}{x+y}+\frac{z+x}{y}\right)^2+2\right)\geq 900\cdot F^2$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that:

$$\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2 a^4+2\right)\cdot\left(\left(\frac{y}{z+x}+\frac{y+z}{x}\right)^2+2\right)\cdot\left(\left(\frac{z}{x+y}+\frac{z+x}{y}\right)^2+2\right)\geq$$

$$\geq\frac{3}{4}(\sqrt{2})^2\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)a^2+\left(\frac{y}{z+x}+\frac{y+z}{z}\right)b^2+\left(\frac{z}{x+y}+\frac{z+x}{y}\right)c^2\right)^2 \quad (1)$$

By Tsintsifas' inequality we have

$$\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\geq 2\sqrt{3}\cdot F \quad (2)$$

Using $AM-GM$ inequality, Cesaro's inequality $(x+y)(y+z)(z+x)\geq 8xyz$ and

Carlitz's inequality $(abc)^{2/3}\geq\frac{4}{3}\sqrt{3}\cdot F$, we obtain:

$$\frac{x+y}{z}a^2+\frac{y+z}{x}b^2+\frac{z+x}{y}c^2\geq$$

$$\geq 3\left(\frac{(x+y)(y+z)(z+x)}{xyz}\cdot(abc)^2\right)^{\frac{1}{3}}\geq 6\left(\frac{4}{3}\sqrt{3}\cdot F\right) \quad (3)$$

By (2) and (3), the inequality (1) becomes

$$\left(\left(\frac{x}{y+z}+\frac{x+y}{z}\right)^2 a^4+2\right)\cdot\left(\left(\frac{y}{z+x}+\frac{y+z}{x}\right)^2+2\right)\cdot\left(\left(\frac{z}{x+y}+\frac{z+x}{y}\right)^2+2\right)\geq$$

$$\geq 3\left(2\sqrt{3}\cdot F+8\sqrt{3}\cdot F\right)^2=900\cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral, $x=y=z$ and $\left(\frac{1}{2}+2\right)a^2=1$,

that is if and only if $a=b=c=\sqrt{\frac{2}{5}}, x=y=z$.

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.