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J.2616 If $m,n,x,y,z>0$, then:

$$\left(\frac{x^2}{(my+nz)^2}+2\right)\left(\frac{y^2}{(mz+nx)^2}+2\right)\left(\frac{z^2}{(mx+ny)^2}+2\right)\geq\frac{27}{(m+n)^2}$$

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Applying Arkady Alt's inequality $(a^2+t^2)(b^2+t^2)(c^2+t^2)\geq\frac{3}{4}t^4(a+b+c)^2$

(with equality if and only if $a=b=c=\frac{t}{\sqrt{2}}$), it follows that

$$\left(\frac{x^2}{(my+nz)^2}+2\right)\left(\frac{y^2}{(mz+nx)^2}+2\right)\left(\frac{z^2}{(mx+ny)^2}+2\right)\geq$$

$$\geq\frac{3}{4}\left(\sqrt{2}\right)^2\left(\frac{x}{my+nz}+\frac{y}{mz+nx}+\frac{z}{mx+ny}\right)^2 \quad (1)$$

Using Bergström inequality and the known inequality:

$(x+y+z)^2\geq 3(xy+xy+zx)$, we obtain:

$$\frac{x}{my+nz}+\frac{y}{mz+nx}+\frac{z}{mx+ny}=\frac{x^2}{mxy+nzx}+\frac{y^2}{myz+nxy}+\frac{z^2}{mzx+nyz}\geq$$

$$\geq\frac{(x+y+z)^2}{(m+n)(xy+yz+zx)}\geq\frac{3}{m+n} \quad (2)$$

By (1) and (2) yields

$$\left(\frac{x^2}{(my+nz)^2}+2\right)\left(\frac{y^2}{(mz+nx)^2}+2\right)\left(\frac{z^2}{(mx+ny)^2}+2\right)\geq 3\left(\frac{3}{m+n}\right)^2=\frac{27}{(m+n)^2}.$$

Equality holds if and only if $x=y=z$ and $\frac{1}{m+n}=1$, that is $m+n=1$.

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.