ROMANIAN MATHEMATICAL MAGAZINE

J.2618 If $x, y, z > 0$ then in any triangle ABC with the area F the following relationship holds:

$$(x^2a^4 + 2(y+z)^2)(y^2b^4 + 2(z+x)^2)(z^2c^4 + 2(x+y)^2) \geq 2304 \cdot (xyz)^2 \cdot F^2$$

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Applying Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$), Cesaro's inequality $(x+y)(y+z)(z+x) \geq 8xyz$

and Tsintsifas' inequality $\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2 \geq 2\sqrt{3}F$, it follows that

$$(x^2a^4 + 2(y+z)^2)(y^2b^4 + 2(z+x)^2)(z^2c^4 + 2(x+y)^2) =$$

$$= (x+y)^2(y+z)^2(z+x)^2\left(\left(\frac{x}{y+z}\right)^2 a^4 + 2\right)\left(\left(\frac{y}{z+x}\right)^2 b^4 + 2\right)\left(\left(\frac{z}{x+y}\right)^2 c^4 + 2\right) \geq$$

$$\geq 64(xyz)^2 \cdot \frac{3}{4}\left(\sqrt{2}\right)^4\left(\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2\right)^2 \geq$$

$$\geq 64(xyz)^2 \cdot 3\left(2\sqrt{3}F\right)^2 = 2304(xyz)^2 \cdot F^2.$$

Equality holds if and only if $a = b = c = \sqrt{2}, x = y = z$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.