J.2619 If $t,u,x,y,z\ge 0$ then in any triangle ABC with the area F the following relationship holds:

$$\frac{(x^2a^4+y^2b^4)h_c}{ta+ub}+\frac{(x^2b^4+y^2c^4)h_a}{tb+uc}+\frac{(x^2c^4+y^2a^4)h_b}{tc+ua}\ge\frac{4\sqrt{3}(x+y)^2}{t+u}\cdot F^2$$

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We have $ah_a=bh_b=ch_c=2F$. Applying Bergström's inequality, it follows that:

$$\frac{(x^2a^4+y^2b^4)h_c}{ta+ub}+\frac{(x^2b^4+y^2c^4)h_a}{tb+uc}+\frac{(x^2c^4+y^2a^4)h_b}{tc+ua}=$$

$$=\frac{(x^2a^4+y^2b^4)ch_c}{tca+ubc}+\frac{(x^2b^4+y^2c^4)ah_a}{tab+uca}+\frac{(x^2c^4+y^2a^4)bh_b}{tbc+uab}=$$

$$=2F\left(\frac{x^2a^4}{tca+ubc}+\frac{x^2b^4}{tab+uca}+\frac{x^2c^4}{tbc+uab}\right)+$$

$$+2F\left(\frac{y^2a^4}{tbc+uab}+\frac{y^2b^4}{tca+ubc}+\frac{y^2c^4}{tab+uca}\right)\ge$$

$$\ge 2F\cdot\frac{x^2(a^2+b^2+c^2)^2}{t(ab+bc+ca)+u(ab+bc+ca)}+2F\cdot\frac{y^2(a^2+b^2+c^2)^2}{t(ab+bc+ca)+u(ab+bc+ca)}=$$

$$=\frac{2F(x^2+y^2)(a^2+b^2+c^2)^2}{(t+u)(ab+bc+ca)}\quad (1)$$

By the known inequality $a^2+b^2+c^2\ge ab+bc+ca$, the inequality

$2(x^2+y^2)\ge(x+y)^2\Leftrightarrow(x-y)^2\ge 0$ and (1), it results that

$$\frac{(x^2a^4+y^2b^4)h_c}{ta+ub}+\frac{(x^2b^4+y^2c^4)h_a}{tb+uc}+\frac{(x^2c^4+y^2a^4)h_b}{tc+ua}\ge$$

$$=\frac{2F(x^2+y^2)(a^2+b^2+c^2)^2}{(t+u)(ab+bc+ca)}\ge\frac{F(x+y)^2(a^2+b^2+c^2)}{t+u}.$$

Using Ionescu-Weitzenbock's inequality $a^2+b^2+c^2\ge 4\sqrt{3}F$, we get:

$$\frac{(x^2a^4+y^2b^4)h_c}{ta+ub}+\frac{(x^2b^4+y^2c^4)h_a}{tb+uc}+\frac{(x^2c^4+y^2a^4)h_b}{tc+ua}\ge\frac{4\sqrt{3}(x+y)^2}{t+u}\cdot F^2.$$

Equality holds if and only if the triangle ABC is equilateral and $x=y$.