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J.2625 If $a, b, c > 0$, then:

$$(a^4+4)(b^4+4)(c^4+4) \ge \frac{4}{3}(a+b+c)^4$$

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Applying Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2) \ge \frac{3}{4}t^4(x+y+z)^2$ for $t=2$ and known inequality $3(a^2+b^2+c^2) \ge (a+b+c)^2$, it follows that

$$(a^4+4)(b^4+4)(c^4+4) \ge \frac{3}{4}(2)^4(a^2+b^2+c^2)^2 \ge \frac{12(a+b+c)^4}{9} = \frac{4(a+b+c)^4}{3}.$$

Equality holds if and only if $a^2=b^2=c^2=\frac{2}{\sqrt{2}}$, that is $a^2=b^2=c^2=\sqrt{2}$.

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \ge \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \ge \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \ge 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \ge \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \ge$$

$$\ge \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.