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J.2626 In any triangle ABC with the semiperimeter s and area F the following inequality holds:

$$(a^2+2s)(b^2+2s)(c^2+2s)\geq 324\cdot F^2$$

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Applying Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$ for $t=\sqrt{2s}$ and inequality $s^2\geq 3F\sqrt{3}$ (item 4.2 from $[1]$) it follows that:

$$(a^2+2s)(b^2+2s)(c^2+2s)\geq\frac{3}{4}\left(\sqrt{2s}\right)^4(a+b+c)^2=\frac{3}{4}(4s^2)(4s^2)=$$

$$=12s^4\geq 12\left(3F\sqrt{3}\right)^2=324\cdot F^2.$$

Equality holds if and only if $a=b=c=\frac{\sqrt{2s}}{\sqrt{2}}$, that is $a=b=c=\frac{3}{2}$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2)\geq$$

$$\geq\frac{3t^2}{4}(t(x+y)+tz)^2=\frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.