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J.2627 In any triangle ABC the following inequality holds:

$$(a^2r_a^2+2)(b^2r_b^2+2)(c^2r_c^2+2)\ge 108\cdot F^2$$

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Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2$

for $t=\sqrt{2}$, it follows that:

$$(a^2r_a^2+2)(b^2r_b^2+2)(c^2r_c^2+2)\ge\frac{3}{4}\left(\sqrt{2}\right)^4(ar_a+br_b+cr_c)^2\quad(1)$$

We have

$$r_a+r_b+r_c=F\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)=$$

$$=\frac{F\big((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\big)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs(3s^2-s(a+b+b+c+c+a)+ab+bc+ca)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs(3s^2-4s^2+s^2+r^2+4Rr)}{F^2}=\frac{sr(4R+r)}{sr}=4R+r\quad(2)$$

Suppose that $a\ge b\ge c$; then $\frac{1}{s-a}\ge\frac{1}{s-b}\ge\frac{1}{s-c}$, hence $r_a\ge r_b\ge r_c$. By Chebyshev's inequality, (2) and Euler inequality ($R\ge 2r$) we get

$$ar_a+br_b+cr_c\ge\frac{1}{3}(a+b+c)(r_a+r_b+r_c)=\frac{2s(4R+r)}{3}\ge 6sr\quad(3)$$

Using (1), (3) and the inequality $s^2\ge 27r^2$ (item 5.11 from $[1]$), yields that

$$(a^2r_a^2+2)(b^2r_b^2+2)(c^2r_c^2+2)\ge 3(ar_a+br_b+cr_c)^2\ge 3(6sr)^2=108\cdot F^2$$

Equality holds if and only if $a=b=c$ and $ar_a=\frac{\sqrt{2}}{\sqrt{2}}=1\Leftrightarrow a=\sqrt{\frac{2}{\sqrt{3}}}$.

$[1]$ O. Bottema, Geometric Inequalities, Groningen 1969

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ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2) \geq \frac{3}{4}t^2((x+y)^2+t^2) \Leftrightarrow \left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}((x+y)^2+t^2)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y)+tz)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.