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J.2628 In any triangle ABC the following inequality holds:

$$\left(a^4r_a^2+2\right)\left(b^4r_b^2+2\right)\left(c^4r_c^2+2\right)\geq 34992\cdot r^6$$

Proposed by D.M.Bătinețu-Giurgiu, Ionuț Ivănescu - Romania

Solution by Titu Zvonaru-Romania

Applying Arkady Alt's inequality $(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$

(with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$), it follows that

$$\left(a^4r_a^2+2\right)\left(b^4r_b^2+2\right)\left(c^4r_c^2+2\right)\geq\frac{3}{4}\left(\sqrt{2}\right)^4\left(a^2r_a+b^2r_b+c^2r_c\right)^2\quad(1)$$

Here are presented two ways to determine the expression $a^2r_a+b^2r_b+c^2r_c$:

$$a^2r_a+b^2r_b+c^2r_c=F\left(\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}\right)=$$

$$=\frac{Fs\left(s^2(a^2+b^2+c^2)-s(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+abc(a+b+c)\right)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs\left(s^2(a^2+b^2+c^2)-s\left((a+b+c)(ab+bc+ca)-3abc\right)+2sabc\right)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{Fs^2\left(s(a^2+b^2+c^2)-2s(ab+bc+ca)+5abc\right)}{F^2}=$$

$$=\frac{s^3\left(a^2+b^2+c^2-2(ab+bc+ca)+20Rr\right)}{F}=$$

$$=\frac{s^3\left(2s^2-2r^2-8Rr-2s^2-2r^2-8Rr+20Rr\right)}{F}=$$

$$=\frac{s^3(-4r^2+4Rr)}{F}=\frac{4s^3r(R-r)}{F}=4s^2(R-r).$$

$$a^2r_a+b^2r_b+c^2r_c=F\left(\frac{a^2}{s-a}+\frac{b^2}{s-b}+\frac{c^2}{s-c}\right)=$$

$$=F\left(\frac{a^2-s^2+s^2}{s-a}+\frac{b^2-s^2+s^2}{s-b}+\frac{c^2-s^2+s^2}{s-c}\right)=$$

$$=F\left(-a-s-b-s-c-s+\frac{s^2}{s-a}+\frac{s^2}{s-b}+\frac{s^2}{s-c}\right)=$$

$$= F\left(-5s + \frac{s^2(3s^2 - s(a+b+b+c+c+a) + ab+bc+ca)}{(s-a)(s-b)(s-c)}\right) =$$

$$= F\left(-5s + \frac{s^3(3s^2 - 4s^2 + s^2 + r^2 + 4Rr)}{s(s-a)(s-b)(s-c)}\right) = F\left(-5s + \frac{s^3 r(r+4R)}{F^2}\right) =$$

$$= F\left(-5s + \frac{s(r+4R)}{r}\right) = -5s^2 r + s^2(r+4R) = 4s^2(R-r) \quad (2)$$

Using $(1), (2)$, Euler's inequality $R \geq 2r$ and the inequality $s^2 \geq 27r^2$ (item 5.11 from $[1]$), it results that

$$(a^4 r_a^2 + 2)(b^4 r_b^2 + 2)(c^4 r_c^2 + 2) \geq \frac{3}{4}(\sqrt{2})^4 (a^2 r_a + b^2 r_b + c^2 r_c)^2 = \left(4s^2(R-r)\right)^2 \geq$$

$$\geq 3(108r^3)^2 = 34992 \cdot r^6.$$

Equality holds if and only if $a = b = c$ and $a^2 r_a = 1$,

that is if and only if $a^3 = b^3 = c^3 = \frac{2}{\sqrt{3}}$.

$[1]$ O. Bottema, *Geometric Inequalities*, Groningen 1969

ARKADY ALT'S INEQUALITY

If $t, x, y, z > 0$ then the following relationship holds:

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4} t^4 (x+y+z)^2$$

with equality if and only if $x = y = z = \frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3}{4} t^2((x+y)^2 + t^2) \Leftrightarrow \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x-y)^2 \geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3t^2}{4}((x+y)^2 + t^2)(t^2 + z^2) \geq$$

$$\geq \frac{3t^2}{4}(t(x+y) + tz)^2 = \frac{3}{4} t^4 (x+y+z)^2.$$

The equality holds if and only if $x = y = z = \frac{t}{\sqrt{2}}$.