RMM - Cyclic Inequalities Marathon 1501 - 1600

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1501. If $a,b,c>0$ such that : $a^2+b^2+c^2=3$ and $\lambda\ge 0,n\ge 0$, then :

$$\frac{a^2}{b^2+\lambda c+n}+\frac{b^2}{c^2+\lambda a+n}+\frac{c^2}{a^2+\lambda b+n}\ge\frac{3}{1+\lambda+n}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{b^2+\lambda c+n}+\frac{b^2}{c^2+\lambda a+n}+\frac{c^2}{a^2+\lambda b+n}$$
$$=\frac{a^4}{a^2b^2+\lambda ca^2+na^2}+\frac{b^4}{b^2c^2+\lambda ab^2+nb^2}+\frac{c^4}{c^2a^2+\lambda bc^2+nc^2}\overset{\text{Bergstrom}}{\ge}$$
$$\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b^2+\lambda\sum_{cyc}ab^2+n\sum_{cyc}a^2}\overset{\text{CBS}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b^2+\lambda.\sqrt{\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)}+n\sum_{cyc}a^2}$$
$$\ge\frac{\left(\sum_{cyc}a^2\right)^2}{\frac{1}{3}\left(\sum_{cyc}a^2\right)^2+\lambda.\sqrt{\frac{1}{3}\left(\sum_{cyc}a^2\right)^2.\left(\sum_{cyc}a^2\right)}+n\sum_{cyc}a^2}$$
$$\overset{a^2+b^2+c^2=3}{=}\frac{3\left(\sum_{cyc}a^2\right)}{\sum_{cyc}a^2+\lambda.\sqrt{\frac{1}{3}\left(\sum_{cyc}a^2\right)^2.3}+n\sum_{cyc}a^2}=\frac{3}{1+\lambda+n}$$
$$\therefore\frac{a^2}{b^2+\lambda c+n}+\frac{b^2}{c^2+\lambda a+n}+\frac{c^2}{a^2+\lambda b+n}\ge\frac{3}{1+\lambda+n}$$
$$\forall\, a,b,c>0\,\middle|\,a^2+b^2+c^2=3\text{ and }\lambda\ge 0,n\ge 0,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1502. If $a,b,c>0$, then prove that :

$$2\sum_{cyc}\frac{a^3c^2}{a+b}\ge\sum_{cyc}a^2b^2$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum_{cyc}\frac{a^3c^2}{a+b}-\sum_{cyc}a^2b^2=2\sum_{cyc}\frac{a^2c^2(a+b-b)}{a+b}-\sum_{cyc}a^2b^2$$
$$=2\sum_{cyc}a^2c^2-2abc\sum_{cyc}\frac{ca}{a+b}-\sum_{cyc}a^2b^2\overset{\text{CBS}}{\ge}$$
$$\sum_{cyc}a^2b^2-2abc.\sqrt{\sum_{cyc}a^2b^2}.\sqrt{\sum_{yc}\frac{1}{(a+b)^2}}\overset{\text{A-G}}{\ge}$$

$$\sum_{cyc} a^2b^2 - 2abc.\sqrt{\sum_{cyc} a^2b^2}.\sqrt{\sum_{yc} \frac{c}{4abc}} \overset{?}{\geq} 0 \Leftrightarrow \sqrt{\sum_{cyc} a^2b^2} \overset{?}{\geq} \frac{2abc}{\sqrt{4abc}}.\sqrt{\sum_{cyc} a}$$

$$\Leftrightarrow \sum_{cyc} a^2b^2 \overset{?}{\geq} abc\sum_{cyc} a \rightarrow \text{true} \therefore 2\sum_{cyc} \frac{a^3c^2}{a+b} \geq \sum_{cyc} a^2b^2$$

$$\forall\, a,b,c > 0,''='' \text{ iff } a = b = c \text{ (QED)}$$

1503. If $a, b, c > 0$ and $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3$, then prove that :

$$(a + 2b)(b + 2c)(c + 2a) \geq 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{a} = A, \sqrt{b} = B, \sqrt{c} = C$ and then : assigning $B + C = x, C + A = y,$ $A + B = z \Rightarrow x + y - z = 2C > 0, y + z - x = 2A > 0$ and $z + x - y = 2B > 0$ $\Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) and $2\sum_{cyc} A = \sum_{cyc} x = 2s$

$$\Rightarrow \sum_{cyc} A = s \rightarrow (1) \Rightarrow A = s - x, B = s - y, C = s - z \therefore ABC = r^2s \rightarrow (2)$$

and such substitutions $\Rightarrow \sum_{cyc} AB = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} AB = 4Rr + r^2 \rightarrow (3),$

$$\sum_{cyc} A^2 = \left(\sum_{cyc} A\right)^2 - 2\sum_{cyc} AB \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} A^2$$

$$= s^2 - 8Rr - 2r^2 \rightarrow (4) \text{ and also, } \sum_{cyc} A^2B^2 = \left(\sum_{cyc} AB\right)^2 - 2ABC\left(\sum_{cyc} A\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s^2 \Rightarrow \sum_{cyc} A^2B^2 = r^2((4R + r)^2 - 2s^2) \rightarrow (5)$$

$$\text{Now, } (a + 2b)(b + 2c)(c + 2a) = (A^2 + 2B^2)(B^2 + 2C^2)(C^2 + 2A^2)$$

$$= 2\sum_{cyc} A^4B^2 + 2\sum_{cyc} A^2B^4 + 9A^2B^2C^2 + 2\sum_{cyc} A^2B^4$$

$$= 2\sum_{cyc}\left(A^2B^2\left(\sum_{cyc} A^2 - C^2\right)\right) + 9A^2B^2C^2 + 2\sum_{cyc} A^2B^4$$

$$\overset{A-G}{\geq} 2\left(\sum_{cyc} A^2\right)\left(\sum_{cyc} A^2B^2\right) + 3A^2B^2C^2 + 6A^2B^2C^2 \overset{?}{\geq} 27 \overset{\sum_{cyc} AB = 3}{=} \left(\sum_{cyc} AB\right)^3$$

$$\overset{\text{via (2),(3),(4) and (5)}}{\Leftrightarrow} 2(s^2 - 8Rr - 2r^2).r^2((4R + r)^2 - 2s^2) + 9r^4s^2 \overset{?}{\geq} (4Rr + r^2)^3$$

$$\Leftrightarrow 4s^4-(32R^2+48Rr+19r^2)s^2+5r(4R+r)^3 \overset{(*)}{\le} 0$$
Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$,
where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$
$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0 \Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$
$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0 \therefore$ in order to prove $(*)$,
it suffices to prove : LHS of $(*)\le 4s^4-4s^2(4R^2+20Rr-2r^2)+4r(4R+r)^3$
$$\Leftrightarrow (16R^2-32Rr+27r^2)s^2 \overset{(**)}{\ge} r(4R+r)^3$$
Again, $(16R^2-32Rr+27r^2)s^2 \overset{\text{Gerretsen}}{\ge} (16R^2-32Rr+27r^2)(16Rr-5r^2)$
$$\overset{?}{\ge} r(4R+r)^3 \Leftrightarrow 48t^3-160t^2+145t-34 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$
$\Leftrightarrow (t-2)\left(32t(t-2)+16t^2+17\right) \overset{?}{\ge} 0 \to$ true $\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (**) \Rightarrow (*)$ is true
$$\therefore (a+2b)(b+2c)(c+2a)\ge 27\ \forall\, a,b,c>0 \,\big|\, \sqrt{ab}+\sqrt{bc}+\sqrt{ca}=3,$$
'' = '' iff $a=b=c=1$ (QED)

1504.

If $a,b,c>0$ and $2(a+b+c)+ab+bc+ca=9$, then prove that :

$$\frac{a+1}{a^2+10a+21}+\frac{b+1}{b^2+10b+21}+\frac{c+1}{c^2+10c+21}\le\frac{3}{16}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a+1}{a^2+10a+21}=\sum_{cyc}\frac{3(a+3)-(a+7)}{2(a+3)(a+7)}\le\frac{3}{16}$$
$$\Leftrightarrow \sum_{cyc}\frac{3}{a+7}-\sum_{cyc}\frac{1}{a+3}\le\frac{3}{8}\Leftrightarrow \frac{3(a+7-a)}{7(a+7)}-\sum_{cyc}\frac{1}{a+3}\le\frac{3}{8}$$
$$\Leftrightarrow \frac{3}{7}\sum_{cyc}\frac{a}{a+7}+\sum_{cyc}\frac{1}{a+3}\overset{(\bullet)}{\ge}\frac{51}{56}$$
$$\text{Now, }\sum_{cyc}\frac{a}{a+7}=\sum_{cyc}\frac{a^2}{a^2+7a}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab+7\sum_{cyc}a}$$
$$\overset{2\sum_{cyc}a+\sum_{cyc}ab=9}{=}\frac{\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a\right)^2-2\left(9-2\sum_{cyc}a\right)+7\sum_{cyc}a}$$
$$\therefore \frac{3}{7}\sum_{cyc}\frac{a}{a+7}\overset{(*)}{\ge}\frac{3t^2}{7(t^2+11t-18)}\left(t=\sum_{cyc}a\right)$$
$$9=2\sum_{cyc}a+\sum_{cyc}ab\overset{\text{Bergstrom}}{\ge}6\sqrt[3]{abc}+3\sqrt[3]{a^2b^2c^2}\Rightarrow x^2+2x-3\le 0$$

$$(x=\sqrt[3]{abc})\Rightarrow(x-1)(x+3)\le 0\Rightarrow x=\sqrt[3]{abc}\le 1\Rightarrow abc\overset{(**)}{\le}1$$

$$\text{Again, }\sum_{cyc}\frac{1}{a+3}=\frac{\sum_{cyc}(b+3)(c+3)}{(a+3)(b+3)(c+3)}=\frac{\sum_{cyc}ab+6\sum_{cyc}a+27}{abc+27+9\sum_{cyc}a+3\sum_{cyc}ab}$$

$$\overset{\substack{2\sum_{cyc}a+\sum_{cyc}ab=9\\ \text{and}\\ \text{via }(**)}}{\ge}\frac{9-2\sum_{cyc}a+6\sum_{cyc}a+27}{1+27+9\sum_{cyc}a+3\left(9-2\sum_{cyc}a\right)}\ \therefore\ \sum_{cyc}\frac{1}{a+3}\overset{(***)}{\ge}\frac{36+4t}{55+3t}$$

$$\therefore(*)+(***)\Rightarrow\frac{3}{7}\sum_{cyc}\frac{a}{a+7}+\sum_{cyc}\frac{1}{a+3}\ge\frac{3t^2}{7(t^2+11t-18)}+\frac{36+4t}{55+3t}\overset{?}{\ge}\frac{51}{56}$$

$$\Leftrightarrow\frac{3t^2(55+3t)+7\left(t^2+11t-18\right)(36+4t)}{7(t^2+11t-18)(55+3t)}\overset{?}{\ge}\frac{51}{56}$$

$$\Leftrightarrow 143t^3+1312t^2-9957t+14202\overset{?}{\ge}0$$

$$\Leftrightarrow(t-3)\left(143t^2+1741(t-3)+489\right)\overset{?}{\ge}0\to\text{true}\because 9-2\sum_{cyc}a=\sum_{cyc}ab$$

$$\le\frac{1}{3}\left(\sum_{cyc}a\right)^2\Rightarrow t^2+6t-27\ge 0\Rightarrow(t+9)(t-3)\ge 0\Rightarrow t\ge 3\Rightarrow(\bullet)\text{ is true}$$

$$\therefore\frac{a+1}{a^2+10a+21}+\frac{b+1}{b^2+10b+21}+\frac{c+1}{c^2+10c+21}\le\frac{3}{16}$$

$\forall\, a,b,c>0\,\big|\,2(a+b+c)+ab+bc+ca=9,''=''$ iff $a=b=c=1$ (QED)

1505. If $x,y,z>0$ and $(x+y)(y+z)(z+x)=64$, then prove that :

$$\frac{x}{(y+z)(x+2y)^2}+\frac{y}{(z+x)(y+2z)^2}+\frac{z}{(x+y)(z+2x)^2}\ge\frac{1}{24}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x}{(y+z)(x+2y)^2}+\frac{y}{(z+x)(y+2z)^2}+\frac{z}{(x+y)(z+2x)^2}$$

$$=\frac{\left(\frac{x}{x+2y}\right)^2}{(xy+zx)}+\frac{\left(\frac{y}{y+2z}\right)^2}{(yz+xy)}+\frac{\left(\frac{z}{z+2x}\right)^2}{(zx+yz)}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\frac{x}{x+2y}\right)^2}{2\sum_{cyc}xy}=\frac{\left(\sum_{cyc}\frac{x^2}{x^2+2xy}\right)^2}{2\sum_{cyc}xy}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x^2+2\sum_{cyc}xy}\right)^2}{2\sum_{cyc}xy}=\frac{\left(\frac{\left(\sum_{cyc}x\right)^2}{\left(\sum_{cyc}x\right)^2}\right)^2}{2\sum_{cyc}xy}\overset{?}{\ge}\frac{1}{24}\Leftrightarrow\sum_{cyc}xy\overset{?}{\le}12$$

$$\overset{(x+y)(y+z)(z+x)=64}{=}\frac{12}{16}\cdot\sqrt[3]{(x+y)^2(y+z)^2(z+x)^2}$$

$$\Leftrightarrow 27\prod_{cyc}(x+y)^2 \overset{?}{\underset{(*)}{\geq}} 64\left(\sum_{cyc} xy\right)^3$$

$$\text{Now, }\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{A-G}{\geq} \left(3.\sqrt[3]{xyz}\right)\left(3.\sqrt[3]{x^2y^2z^2}\right) = 9xyz$$

$$\Rightarrow 9\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - xyz\right) \geq 8\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$\Rightarrow 9\prod_{cyc}(x+y) \geq 8\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$\Rightarrow 81\prod_{cyc}(x+y)^2 \geq 64\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} xy\right)^2 \geq 64\left(3\sum_{cyc} xy\right)\left(\sum_{cyc} xy\right)^2$$

$$\Rightarrow 27\prod_{cyc}(x+y)^2 \geq 64\left(\sum_{cyc} xy\right)^3 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{x}{(y+z)(x+2y)^2} + \frac{y}{(z+x)(y+2z)^2} + \frac{z}{(x+y)(z+2x)^2} \geq \frac{1}{24}$$

$$\forall\, x,y,z > 0 \,\big|\, (x+y)(y+z)(z+x) = 64, '' ='' \text{ iff } x = y = z = 2 \text{ (QED)}$$

1506. If $a,b,c > 1$ and $a+b+c = 6$, then prove that :

$$\frac{a}{a^2-a+1} + \frac{b}{b^2-b+1} + \frac{c}{c^2-c+1} \geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a-1 = x, b-1 = y, c-1 = z \text{ and then : } \sum_{cyc}\frac{a}{a^2-a+1}$$

$$= \sum_{cyc}\frac{a}{(a-1)^2+a} = \sum_{cyc}\frac{x+1}{x^2+x+1} = \sum_{cyc}\frac{x+1+x^2-x^2}{x^2+x+1}$$

$$= 3 - \sum_{cyc}\frac{x^2}{x^2+x+1} \overset{A-G}{\geq} 3 - \sum_{cyc}\frac{x^2}{2x+x} = 3 - \frac{1}{3}\sum_{cyc} x = 3 - \frac{1}{3}.3$$

$$\left(\because \sum_{cyc} x = \sum_{cyc}(a-1) \overset{a+b+c=6}{=} 6-3 = 3\right) = 2$$

$$\therefore \frac{a}{a^2-a+1} + \frac{b}{b^2-b+1} + \frac{c}{c^2-c+1} \geq 2\ \forall\, a,b,c > 1 \,\Big|\, \sum_{cyc} a = 3,$$

$'' ='' \text{ iff } a = b = c = 2$ (QED)

1507. If $a, b, c > \frac{1}{3}$ and $a + b + c = 3$, then prove that :

$$\frac{1}{3a^2 - 3a + 1} + \frac{1}{3b^2 - 3b + 1} + \frac{1}{3c^2 - 3c + 1} \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3a^2 - 3a + 1} = \sum_{cyc} \frac{3}{9a^2 - 9a + 3} = \sum_{cyc} \frac{3}{(3a-1)^2 - (3a-1) + 1}$$

$$= \sum_{cyc} \frac{3}{x^2 - x + 1} \ (x = 3a - 1, y = 3b - 1, z = 3c - 1 \text{ and } x, y, z > 0) \geq 3$$

$$\Leftrightarrow \sum_{cyc} (y^2 - y + 1)(z^2 - z + 1) \geq \prod_{cyc} (x^2 - x + 1)$$

$$\Leftrightarrow xyz \sum_{cyc} xy + xyz + \sum_{cyc} x^2 + 2 - x^2y^2z^2 - xyz \sum_{cyc} x - \sum_{cyc} x \overset{(*)}{\geq} 0$$

$$\text{Now}, \because \sum_{cyc} x = \sum_{cyc} (3a - 1) \overset{a+b+c=3}{=} 6 \therefore \text{LHS of } (*) \overset{A-G}{\geq}$$

$$3(xyz)^{\frac{5}{3}} + xyz + \frac{1}{3}.36 + 2 - x^2y^2z^2 - 6xyz \sum_{cyc} x - 6 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^5 - 5t^3 + 8 - t^6 \overset{?}{\geq} 0 \ (t = \sqrt[3]{xyz}) \Leftrightarrow t^6 - 3t^5 + 5t^3 - 8 \overset{?}{\leq} 0$$

$$\Leftrightarrow (t-2)(t^5 - t^4 - 2t^3 + t^2 + 2t + 4) \underbrace{\overset{?}{\leq}}_{(**)} 0$$

$$\text{Now}, t^4 - t^3 - 2t^2 + t + 2 = (t^4 - 2t^2 + 1) - t(t^2 - 1) + 1$$

$$= (t^2 - 1)^2 - (t - 1 + 1)(t^2 - 1) + 1 = (t^2 - 1)^2 - (t-1)^2(t+1) - t^2 + 2$$

$$= (t-1)^2(t^2 + t) - t^2 + 2 = t(t-1)^2 + t^2(t^2 - 2t + 1) - t^2 + 2$$

$$= t(t-1)^2 + \frac{1}{16}(16t^4 - 32t^3 + 32)$$

$$= t(t-1)^2 + \frac{1}{16}\left((4t^2 + 4t + 3)(2t-3)^2 + 5\right) > 0$$

$$\Rightarrow t^4 - t^3 - 2t^2 + t + 2 > 0 \rightarrow (1)$$

$$\text{Again}, t = \sqrt[3]{xyz} \overset{A-G}{\leq} \frac{1}{3} \sum_{cyc} x = \frac{1}{3}.6 \Rightarrow t - 2 \leq 0 \rightarrow (2) \therefore (1), (2) \Rightarrow \text{LHS of } (**) =$$

$$(t-2)\left(t(t^4 - t^3 - 2t^2 + t + 2) + 4\right) \leq 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{3a^2 - 3a + 1} + \frac{1}{3b^2 - 3b + 1} + \frac{1}{3c^2 - 3c + 1} \geq 3 \ \forall a, b, c > \frac{1}{3} \mid \sum_{cyc} a = 3,$$

$'' ='' \text{ iff } a = b = c = 1$ (QED)

1508. If $a,b,c>2$ and $a+b+c=9$, then prove that :

$$\frac{1}{a^2-4a+5}+\frac{1}{b^2-4b+5}+\frac{1}{c^2-4c+5}\ge\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a-2=x, b-2=y, c-2=z \text{ and then : } \sum_{cyc}\frac{1}{a^2-4a+5}$$

$$=\sum_{cyc}\frac{1}{(a-2)^2+1}=\sum_{cyc}\frac{1}{x^2+1}=\sum_{cyc}\frac{1+x^2-x^2}{x^2+1}=3-\sum_{cyc}\frac{x^2}{x^2+1}\overset{A-G}{\ge}$$

$$3-\sum_{cyc}\frac{x^2}{2x}=3-\frac{1}{2}\sum_{cyc}x=3-\frac{1}{2}.3\left(\because\sum_{cyc}x=\sum_{cyc}(a-2)\overset{a+b+c=9}{=}9-6=3\right)$$

$$=\frac{3}{2}\therefore\frac{1}{a^2-4a+5}+\frac{1}{b^2-4b+5}+\frac{1}{c^2-4c+5}\ge\frac{3}{2}\ \forall a,b,c>2\,\Big|\sum_{cyc}a=9,$$

$$''='' \text{ iff } a=b=c=3 \text{ (QED)}$$

1509. If $a,b,c>\frac{4}{3}$ and $a+b+c=6$, then prove that :

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\ge\frac{6}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{a^2+1}=\sum_{cyc}\frac{9a}{9a^2+9}=\sum_{cyc}\frac{3(3a-4+4)}{9a^2-24a+16+24a-7}$$

$$=\sum_{cyc}\frac{3(3a-4+4)}{(3a-4)^2+8(3a-4)+25}$$

$$=\sum_{cyc}\frac{3(x+4)}{x^2+8x+25}\ (x=3a-4>0 \text{ and analogs})\ge\frac{6}{5}$$

$$\Leftrightarrow\sum_{cyc}\frac{x+4}{x^2+8x+25}\overset{(*)}{\ge}\frac{2}{5}$$

$$\text{Now, }\frac{x+4}{x^2+8x+25}\overset{?}{\ge}\frac{12-x}{75}\Leftrightarrow 75(x+4)\overset{?}{\ge}(12-x)(x^2+8x+25)$$

$$\left(\sum_{cyc}x=\sum_{cyc}(3a-4)\overset{a+b+c=6}{=}6\Rightarrow x<6<12\Rightarrow 12-x>0\right)$$

$$\Leftrightarrow x(x-2)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \frac{x+4}{x^2+8x+25} \geq \frac{12-x}{75} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{x+4}{x^2+8x+25} \geq \sum_{cyc} \frac{12-x}{75} = \frac{36-\sum_{cyc} x}{75} = \frac{36-6}{75} = \frac{2}{5} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \geq \frac{6}{5}\ \forall a,b,c > \frac{4}{3} \mid \sum_{cyc} a = 6,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1510. If $x, y > 0$ and $x + y + xy = 3$, then prove that :

$$\sqrt{9-x^2} + \sqrt{9-y^2} + \frac{x+y}{4} \leq \frac{1+8\sqrt{2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumva Chakraborty-Kolkata-India

$$xy = 3-(x+y) \leq \frac{(x+y)^2}{4} \Rightarrow t^2 \geq 12-4t\ (t = x+y)$$

$$\Rightarrow t^2+4t-12 \geq 0 \Rightarrow (t+6)(t-2) \geq 0 \Rightarrow t \geq 2 \to (1)$$

$$\text{Now, } \sqrt{9-x^2}+\sqrt{9-y^2} \overset{CBS}{\leq} \sqrt{2}.\sqrt{18-((x+y)^2-2xy)} \overset{x+y+xy=3}{=}$$

$$\sqrt{2}.\sqrt{18-\left((x+y)^2-2\big(3-(x+y)\big)\right)} = \sqrt{2}.\sqrt{18-(t^2-6+2t)}$$

$$= \sqrt{2}.\sqrt{(4-t)(t+6)} = \sqrt{2}.\sqrt{2(4-t).\left(\frac{t+6}{2}\right)} \overset{A-G}{\leq} \sqrt{2}.\left(\frac{8-2t+\frac{t+6}{2}}{2}\right)$$

$$\left(\text{note} : 2 \overset{via\ (1)}{\leq} t = 3-xy < 3 < 4\right) = \sqrt{2}.\frac{22-3t}{4} \Rightarrow \sqrt{9-x^2}+\sqrt{9-y^2}+\frac{x+y}{4}$$

$$\leq \sqrt{2}.\frac{22-3t}{4}+\frac{t}{4} = \frac{11\sqrt{2}}{2} - \frac{(3\sqrt{2}-1)t}{4} \overset{via\ (1)}{\leq} \frac{11\sqrt{2}}{2} - \frac{3\sqrt{2}-1}{2}$$

$$\therefore \sqrt{9-x^2}+\sqrt{9-y^2}+\frac{x+y}{4} \leq \frac{1+8\sqrt{2}}{2}\ \forall x,y > 0 \mid x+y+xy = 3,$$

$$'' ='' \text{ iff } x = y = 1 \text{ (QED)}$$

1511. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then prove that :

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{c^2+a^2} + \frac{c^3}{a^2+b^2} \geq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a^3}{b^2+c^2} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} \frac{a^2}{b^2+c^2}\right)\left(\sum_{cyc} a\right)$$

$$\left(\text{WLOG assuming } a \geq b \geq c \Rightarrow \frac{a^2}{b^2+c^2} \geq \frac{b^2}{c^2+a^2} \geq \frac{c^2}{a^2+b^2}\right)$$

$$= \frac{1}{3}\left(\sum_{cyc}\frac{a^4}{a^2b^2+c^2a^2}\right)\left(\sum_{cyc}a\right) \overset{\text{Bergstrom}}{\geq} \frac{1}{3}\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{2\sum_{cyc}a^2b^2}\cdot\sum_{cyc}a \overset{a^2+b^2+c^2=3}{=}$$

$$\frac{1}{\sqrt{3}}\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{2\sum_{cyc}a^2b^2}\cdot\frac{\sum_{cyc}a}{\sqrt{\sum_{cyc}a^2}} \overset{?}{\geq} \frac{3}{2} \Leftrightarrow \left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2\right)^3 \boxed{\overset{?}{\underset{(*)}{\geq}}} 27\left(\sum_{cyc}a^2b^2\right)^2$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}a = \sum_{cyc}x = 2s \Rightarrow \sum_{cyc}a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$$\therefore abc = r^2s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc}ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc}ab = 4Rr + r^2 \to (3), \sum_{cyc}a^2 = \left(\sum_{cyc}a\right)^2 - 2\sum_{cyc}ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc}a^2 = s^2 - 8Rr - 2r^2 \to (4),$$

$$\sum_{cyc}a^2b^2 = \left(\sum_{cyc}ab\right)^2 - 2abc\left(\sum_{cyc}a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc}a^2b^2 = r^2\left((4R+r)^2 - 2s^2\right) \to (5)$$

$$\text{Via (1), (4), (5), } (*) \Leftrightarrow s^2\left(s^2 - 8Rr - 2r^2\right)^3 \geq 27r^4\left((4R+r)^2 - 2s^2\right)^2$$

$$\Leftrightarrow s^8 - (24Rr - 6r^2)s^6 + r^2(192R^2 + 96Rr - 96r^2)s^4$$

$$-r^3(512R^3 - 1344R^2r - 768Rr^2 - 100r^3)s^2 - 27r^4(4R+r)^4 \boxed{\overset{(**)}{\geq}} 0 \text{ and}$$

$$\because \left(s^2 - 16Rr + 5r^2\right)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\text{LHS of } (**) \geq \left(s^2 - 16Rr + 5r^2\right)^4$$

$$\Leftrightarrow (20R - 13r)s^6 - r(672R^2 - 528Rr + 123r^2)s^4$$

$$+r^2(7936R^3 - 7008R^2r + 2784Rr^2 - 200r^3)s^2$$

$$-r^3(36224R^4 - 37504R^3r + 20496R^2r^2 - 3784Rr^3 + 326r^4) \boxed{\overset{(***)}{\geq}} 0$$

$$\text{and} \because (20R - 13r)\left(s^2 - 16Rr + 5r^2\right)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (20R - 13r)\left(s^2 - 16Rr + 5r^2\right)^3$$

$$\Leftrightarrow (288R^2 - 396Rr + 72r^2)s^4 - r(7424R^3 - 12576R^2r + 4956Rr^2 - 775r^3)s^2$$

$$+r^2(45696R^4 - 92544R^3r + 53424R^2r^2 - 14316Rr^3 + 1299r^4) \boxed{\overset{(****)}{\geq}} 0$$

$$and \because (288R^2-396Rr+72r^2)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (***), \text{it suffices to prove : LHS of } (****) \geq$$
$$(288R^2-396Rr+72r^2)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (1792R^3-2976R^2r+1308Rr^2+55r^3)s^2 \boxed{\overset{(*****)}{\geq}}$$
$$r(28032R^4-54912R^3r+35568R^2r^2-7104Rr^3+501r^4)$$
$$\text{Now, LHS of } (*****) \overset{Gerretsen}{\geq} (1792R^3-2976R^2r+1308Rr^2+55r^3)\binom{16Rr}{-5r^2}$$
$$\overset{?}{\geq} r(28032R^4-54912R^3r+35568R^2r^2-7104Rr^3+501r^4)$$
$$\Leftrightarrow 160t^4-416t^3+60t^2+361t-194 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(160t^2+224t+316)+729\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2$$
$$\Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2} \geq \frac{3}{2} \ \forall a,b,c>0 \,|\, a^2+b^2+c^2=3,$$
$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1512. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\frac{a^3}{9-a^2}+\frac{b^3}{9-b^2}+\frac{c^3}{9-c^2} \geq \frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{9-a^2}=\sum_{cyc}\frac{a^3}{(3-a)(3+a)} \overset{a+b+c=3}{=} \sum_{cyc}\frac{a^3}{(b+c)\left((a+b)+(c+a)\right)}$$
$$\overset{Holder}{\geq} \frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}\left((b+c)(a+b)\right)+3\sum_{cyc}\left((b+c)(c+a)\right)}$$
$$=\frac{\left(\sum_{cyc}a\right)^3}{3\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)+3\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)}=\frac{\left(\sum_{cyc}a\right)^3}{6\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)}$$
$$\geq \frac{\left(\sum_{cyc}a\right)^3}{6\left(\left(\sum_{cyc}a\right)^2+\frac{\left(\sum_{cyc}a\right)^2}{3}\right)}=\frac{3}{8} \therefore \frac{a^3}{9-a^2}+\frac{b^3}{9-b^2}+\frac{c^3}{9-c^2} \geq \frac{3}{8}$$
$$\forall a,b,c>0 \,|\, a+b+c=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1513.

If $x,y,z \in \mathbb{R}$ and $x^2+y^2+z^2=3$, then prove that :

$$8(2-x)(2-y)(2-z) \geq (x+yz)(y+zx)(z+xy)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case : exactly two among } x,y,z \text{ equal zero}}$ and then : LHS $>$

$$8\left(2-\sqrt{3}\right)^3 \left(\begin{matrix} \because x^2,y^2,z^2 < 3 \\ \therefore -\sqrt{3} < x,y,z < \sqrt{3} \end{matrix} \Rightarrow (2-x),(2-y),(2-z) > 2-\sqrt{3}\right) > 0$$

$$= \text{RHS}$$

$\boxed{\text{Case : exactly one among } x,y,z \text{ equals zero}}$ WLOG we may assume $x=0$ and so, $y,z \neq 0$ with $y^2+z^2=3$ and we are to prove : $16(2-y)(2-z) \geq y^2z^2 \to (1)$

$$\boxed{\text{Sub-case : } yz < 0} \therefore \begin{Bmatrix} y>0, z<0 \\ \text{or} \\ y<0, z>0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \because y^2,z^2 < 3 \\ \therefore -\sqrt{3} < y,z < \sqrt{3} \end{Bmatrix} \therefore \text{LHS of } (1) >$$

$$32\left(2-\sqrt{3}\right)\left(\because \min\{(2-y),(2-z)\} > 2-\sqrt{3} \text{ and } \max\{(2-y),(2-z)\} > 2\right)$$

$$\approx 8.574374 > \frac{9}{4} \geq y^2z^2 \left(\because 3 = |y|^2+|z|^2 \overset{A-G}{\geq} 2|yz| \Rightarrow |yz| \leq \frac{3}{2}\right)$$

$$\Rightarrow (1) \text{ is true (strict inequality)}$$

$$\boxed{\text{Sub-case : } yz > 0} \text{ and } (1) \Leftrightarrow 64 - 32(y+z) + 16yz \geq y^2z^2$$

$$\Leftrightarrow 64 + yz(16-yz) \geq 32(y+z) \to (a)$$

Now, $0 < yz = |yz| \leq \frac{3}{2} < 16 \therefore yz(16-yz) > 0 \Rightarrow$ LHS of (i) > 64 and if $y+z<0$, then (a) is definitely true and so, we focus on : $y+z>0$ and then :

$$(a) \overset{y^2+z^2=3}{\Leftrightarrow} \left(64+t(16-t)\right)^2 \geq 1024(3+2t)\ (t=yz) \Leftrightarrow$$

$$t^4 - 32t^3 + 128t^2 + 1024 \geq 0 \Leftrightarrow (t-6)\left(t^3-26t^2-28t-168\right)+16 \geq 0 \text{ and}$$

$$\because t \leq \frac{3}{2} < 6 \therefore \text{it suffices to prove : } t^3-26t^2-28t-168 < 0$$

$$\Leftrightarrow 4t^3 - 104t^2 - 112t - 672 < 0 \Leftrightarrow (t-23)(2t-3)^2 - 397t - 465 < 0 \to true$$

$$\because 0 < t \leq \frac{3}{2} < 23 \Rightarrow (a) \Rightarrow (1) \text{ is true (strict inequality)}$$

We now focus on the cases when none of x,y,z equals zero

$$\boxed{\text{Case : } x,y,z<0} \text{ and then : } 8(2-x)(2-y)(2-z) > 8\left(2^3\right) = 64 \to (2)$$

$$\text{and } (x+yz)(y+zx)(z+xy) = xyz + x^2y^2z^2 + xyz\sum_{cyc} x^2 + \sum_{cyc} x^2y^2$$

$$\overset{x^2+y^2+z^2=3}{=} 4xyz + x^2y^2z^2 + \sum_{cyc} x^2y^2 \leq 4xyz + x^2y^2z^2 + \frac{1}{3}\left(\sum_{cyc} x^2\right)^2$$

$$\overset{\substack{x^2+y^2+z^2=3 \\ \text{and} \\ \because xyz<0}}{<} x^2y^2z^2 + 3 \leq 1+3$$

$$\left(\because 3 = |x|^2+|y|^2+|z|^2 \overset{A-G}{\geq} 3\sqrt[3]{|xyz|^2} \Rightarrow |xyz|^2 = x^2y^2z^2 \leq 1\right) = 4 < 64$$

$$\overset{\text{via } (2)}{<} 8(2-x)(2-y)(2-z)$$

$$\therefore 8(2-x)(2-y)(2-z) > (x+yz)(y+zx)(z+xy)$$

$\boxed{\text{Case : exactly two among } x,y,z<0}$ and WLOG we may assume $y,z<0$ and $\because$

$$0 < x < \sqrt{3} \therefore 8(2-x)(2-y)(2-z) > 8\left(2-\sqrt{3}\right)\left(2^2\right) = 32\left(2-\sqrt{3}\right) \to (3)$$

$$\text{and } (x+yz)(y+zx)(z+xy)=4xyz+x^2y^2z^2+\sum_{cyc}x^2y^2\le$$

$$4xyz+x^2y^2z^2+\frac{1}{3}\left(\sum_{cyc}x^2\right)^2\overset{x^2+y^2+z^2=3}{\le}4xyz+x^2y^2z^2+3\overset{x^2y^2z^2\le 1}{\le}$$

$$4+1+3\left(\begin{matrix}\because xyz>0\ \therefore x^2y^2z^2\le 1\\ \Rightarrow xyz\le 1\end{matrix}\right)\Rightarrow (x+yz)(y+zx)(z+xy)<8\to(4)$$

$$\therefore (3),(4)\Rightarrow \text{it suffices to prove}: 32(2-\sqrt{3})>8\Leftrightarrow 7>4\sqrt{3}\Leftrightarrow 49>48$$

$$\therefore 8(2-x)(2-y)(2-z)>(x+yz)(y+zx)(z+xy)$$

$$\boxed{Case: exactly\ one\ among\ x,y,z<0}\ \text{and LHS}=8\left(8-xyz+2\sum_{cyc}xy-4\sum_{cyc}x\right)$$

$$\overset{x^2+y^2+z^2=3}{=}8\left(5+\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)-xyz-4\sum_{cyc}x\right)$$

$$=8\left(\left(\sum_{cyc}x\right)^2-4\sum_{cyc}x+4+1-xyz\right)=8\left(\sum_{cyc}x-2\right)^2+8-8xyz\overset{xyz<0}{>}8$$

$$\therefore \text{LHS}>8\to(5)\text{ and }(x+yz)(y+zx)(z+xy)\le 4xyz+x^2y^2z^2+3\overset{\substack{x^2y^2z^2\le 1\\ and\\ xyz<0}}{\le}4$$

$$<8\overset{\text{via }(5)}{<}\text{LHS}\ \therefore 8(2-x)(2-y)(2-z)>(x+yz)(y+zx)(z+xy)$$

$$\boxed{Case: x,y,z>0}\ \text{and then}: 8(2-x)(2-y)(2-z)\ge(x+yz)(y+zx)(z+xy)$$

$$\Leftrightarrow 64+16\sum_{cyc}xy-32\sum_{cyc}x\ge 12xyz+x^2y^2z^2+\sum_{cyc}x^2y^2$$

$$\overset{x^2+y^2+z^2=3}{\Leftrightarrow}\frac{64}{3}\sum_{cyc}x^2++16\sum_{cyc}xy\ge$$

$$\frac{12\sqrt{3}xyz}{\sqrt{\sum_{cyc}x^2}}+\frac{32(\sum_{cyc}x).\sqrt{\sum_{cyc}x^2}}{\sqrt{3}}+\frac{9x^2y^2z^2}{(\sum_{cyc}x^2)^2}+\frac{3\sum_{cyc}x^2y^2}{\sum_{cyc}x^2}$$

$$\Leftrightarrow\frac{64\sum_{cyc}x^2+48\sum_{cyc}xy}{3}-\frac{9x^2y^2z^2+3(\sum_{cyc}x^2)(\sum_{cyc}x^2y^2)}{(\sum_{cyc}x^2)^2}$$

$$\ge\frac{36xyz+32(\sum_{cyc}x)(\sum_{cyc}x^2)}{\sqrt{3\sum_{cyc}x^2}}$$

$$\Leftrightarrow\frac{\left((64\sum_{cyc}x^2+48\sum_{cyc}xy)(\sum_{cyc}x^2)^2-27x^2y^2z^2-9(\sum_{cyc}x^2)(\sum_{cyc}x^2y^2)\right)^2}{9(\sum_{cyc}x^2)^4}$$

$$\overset{(*)}{\geq}\frac{\left(36xyz+32\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)\right)^2}{3\sum_{cyc}x^2}$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\rightarrow(\bullet)$

$\Rightarrow x=s-X, y=s-Y, z=s-Z\therefore xyz=r^2s\rightarrow(\bullet\bullet)$ and such substitutions $\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)\Rightarrow\sum_{cyc}xy=4Rr+r^2\rightarrow(\bullet\bullet\bullet)$

$$\text{and }\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via }(\bullet)\text{ and }(\bullet\bullet\bullet)}{=}s^2-2(4Rr+r^2)$$

$\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\rightarrow(\bullet\bullet\bullet\bullet)$ and also, $\sum_{cyc}x^2y^2$

$$=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)\overset{\text{via }(\bullet),(\bullet\bullet)\text{ and }(\bullet\bullet\bullet)}{=}(4Rr+r^2)^2-2r^2s.s\Rightarrow\sum_{cyc}x^2y^2$$

$=r^2\left((4R+r)^2-2s^2\right)\rightarrow(\bullet\bullet\bullet\bullet\bullet)\therefore(\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)$ and $(\bullet\bullet\bullet\bullet\bullet)\Rightarrow(*)\Leftrightarrow$

$$\frac{\left(\begin{matrix}\left(64(s^2-8Rr-2r^2)+48(4Rr+r^2)\right)(s^2-8Rr-2r^2)^2-27r^4s^2\\-9(s^2-8Rr-2r^2)\left(r^2\left((4R+r)^2-2s^2\right)\right)\end{matrix}\right)^2}{9(s^2-8Rr-2r^2)^4}$$

$$\geq\frac{\left(36r^2s+32s(s^2-8Rr-2r^2)\right)^2}{3(s^2-8Rr-2r^2)}$$

$$\Leftrightarrow 256s^{12}-(12288Rr+4224r^2)s^{10}+r^2(250368R^2+163776Rr+23541r^2)s^8$$
$$-r^3(2782720R^3+2572176R^2r+724152Rr^2+64000r^3)s^6$$
$$r^4(17835072R^4+20546880R^3r+8418096R^2r^2+1473144Rr^3+93714r^4)s^4$$
$$-r^5\left(\begin{matrix}62505984R^5+83818752R^4r+43993920R^3r^2\\+11329968R^2r^3+1433976Rr^4+71400r^5\end{matrix}\right)s^2$$
$$+r^6\left(\begin{matrix}93392896R^6+140089344R^5r+87555840R^4r^2\\+29185280R^3r^3+5472240R^2r^4+547224Rr^5+22801r^6\end{matrix}\right)\overset{(**)}{\geq}0$$

Now, $688768t^3-1977828t^2+1834338t-557705\ \left(t=\frac{R}{r}\right)$

$$=(t-2)(688768t^2-600292t+633754)+709803\overset{\text{Euler}}{\geq}709803>0,$$
$$1073220t^4-4106172t^3+5683266t^2-3426096t+766554$$
$$=(t-2)\left(\begin{matrix}1073220t^3-1959732t^2\\+1763802t+101508\end{matrix}\right)+969570\overset{\text{Euler}}{\geq}969570>0\text{ and}$$
$$3574848t^5-17142456t^4+31580580t^3-28406220t^2+12606927t$$
$$-2223690=(t-2)\left(\begin{matrix}(t-2)\left(\begin{matrix}3574848t^3-2843064t^2\\+5908932t+6601764\end{matrix}\right)\\+15378255\end{matrix}\right)+2125764$$

$$\overset{\text{Euler}}{\geq} 2125764 > 0 \therefore P = 256(s^2-16Rr+5r^2)^6$$
$$+(12288Rr-11904r^2)(s^2-16Rr+5r^2)^5$$
$$+r^2(250368R^2-481344Rr+225141r^2)(s^2-16Rr+5r^2)^4$$
$$+4r^3(688768R^3-1977828R^2r+1834338Rr^2-557705r^3)(s^2-16Rr+5r^2)^3$$
$$+16r^4\begin{pmatrix}1073220R^4-4106172R^3r+5683266R^2r^2\\-3426096Rr^3+766554r^4\end{pmatrix}(s^2-16Rr+5r^2)^2+$$
$$16r^5\begin{pmatrix}3574848R^5-17142456R^4r+31580580R^3r^2\\-28406220R^2r^3+12606927Rr^4-2223690r^5\end{pmatrix}(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$
$$\boxed{\text{LHS of } (**) \geq P}$$
$$\Leftrightarrow 2476160t^6-14330976t^5+33045330t^4-39546092t^3$$
$$+26192019t^2-9169998t+1332368 \geq 0$$
$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}262992t^4+2213168t^3(t-2)+5435346t^2\\-99364t+4053179\end{pmatrix}+7440174\right)$$
$$\geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore 8(2-x)(2-y)(2-z) \geq (x+yz)(y+zx)(z+xy)\ \forall\, x,y,z > 0$$
$$\text{and so, combining } all \text{ cases}, 8(2-x)(2-y)(2-z) \geq (x+yz)(y+zx)(z+xy)$$
$$\forall\, x,y,z \in \mathbb{R} \,\middle|\, x^2+y^2+z^2=3, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

1514.

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{b\sqrt{b}(b^5+c^5)+a\sqrt{a}(a^5+c^5)}{\sqrt{c}(a+b)}+\frac{b\sqrt{b}(a^5+b^5)+c\sqrt{c}(a^5+c^5)}{\sqrt{a}(b+c)}$$
$$+\frac{a\sqrt{a}(a^5+b^5)+c\sqrt{c}(b^5+c^5)}{\sqrt{b}(a+c)} \geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = (b^5+c^5)\left(\frac{b\sqrt{b}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{b}(c+a)}\right)$$
$$+(c^5+a^5)\left(\frac{a\sqrt{a}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{a}(b+c)}\right)+(a^5+b^5)\left(\frac{b\sqrt{b}}{\sqrt{a}(b+c)}+\frac{a\sqrt{a}}{\sqrt{b}(a+c)}\right) \rightarrow (1)$$
$$\text{Now, } \frac{b\sqrt{b}}{\sqrt{c}(a+b)}+\frac{c\sqrt{c}}{\sqrt{b}(c+a)}=\frac{b^2}{\sqrt{bc}(a+b)}+\frac{c^2}{\sqrt{bc}(c+a)}$$
$$\overset{\text{Bergstrom}}{\geq} \frac{(b+c)^2}{\sqrt{bc}((c+a)+(a+b))} \text{ and analogs} \rightarrow (2) \therefore (1) \text{ and } (2) \Rightarrow$$
$$\text{LHS} \geq \sum_{\text{cyc}}\left((b^5+c^5).\frac{(b+c)^2}{\sqrt{bc}((c+a)+(a+b))}\right) \overset{A-G}{\geq}$$

$$3.\sqrt[3]{\left(\prod_{cyc}(b^5+c^5)\right)\frac{\prod_{cyc}(b+c)^2}{abc\prod_{cyc}((c+a)+(a+b))}}$$

$$\overset{Cesaro}{\geq} 3.\sqrt[3]{8(abc)^5.\frac{\prod_{cyc}(b+c)^2}{abc\prod_{cyc}((c+a)+(a+b))}}$$

$$\overset{abc=1}{=} 6.\sqrt[3]{\frac{\prod_{cyc}(b+c)^2}{abc\prod_{cyc}((c+a)+(a+b))}} \overset{?}{\geq} 6$$

$$\Leftrightarrow \prod_{cyc}(a+b)^2 \underset{(*)}{\overset{?}{\geq}} abc\prod_{cyc}((c+a)+(a+b))$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s \Rightarrow \sum_{cyc}a=s \Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=(s-x)(s-y)(s-z) \Rightarrow abc=r^2s$ and via such substitutions, $(*) \Leftrightarrow$
$x^2y^2z^2 \geq r^2s(x+y)(y+z)(z+x) \Leftrightarrow 16R^2r^2s^2 \geq r^2s.2s(s^2+2Rr+r^2)$
$\Leftrightarrow s^2 \leq 8R^2-2Rr-r^2 \Leftrightarrow s^2-4R^2-4Rr-3r^2-2(2R+r)(R-2r) \leq 0 \rightarrow$ true
$\because s^2-4R^2-4Rr-3r^2 \overset{Gerretsen}{\leq} 0$ and $-2(2R+r)(R-2r) \overset{Euler}{\leq} 0 \Rightarrow (*)$ is true

$$\therefore \frac{b\sqrt{b}(b^5+c^5)+a\sqrt{a}(a^5+c^5)}{\sqrt{c}(a+b)}+\frac{b\sqrt{b}(a^5+b^5)+c\sqrt{c}(a^5+c^5)}{\sqrt{a}(b+c)}$$
$$+\frac{a\sqrt{a}(a^5+b^5)+c\sqrt{c}(b^5+c^5)}{\sqrt{b}(a+c)} \geq 6\ \forall\, a,b,c>0 \,\big|\, abc=1,$$
$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1515.

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{\sqrt{a^3+b^3}\left(\sqrt[3]{b}.\sqrt[4]{b^5+c^5}+\sqrt[3]{a}.\sqrt[4]{c^5+a^5}\right)}{\sqrt[3]{bc}.\sqrt{b^3+c^3}+\sqrt[3]{ac}.\sqrt{c^3+a^3}}$$
$$+\frac{\sqrt{b^3+c^3}\left(\sqrt[3]{b}.\sqrt[4]{a^5+b^5}+\sqrt[3]{c}.\sqrt[4]{c^5+a^5}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{ac}.\sqrt{c^3+a^3}}$$
$$+\frac{\sqrt{c^3+a^3}\left(\sqrt[3]{a}.\sqrt[4]{a^5+b^5}+\sqrt[3]{c}.\sqrt[4]{b^5+c^5}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{bc}.\sqrt{b^3+c^3}} \geq 3.\sqrt[4]{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A,B,C>0,(A+B),(B+C),(C+A)\text{ form sides of } a \text{ triangle}$$
$$(\because (A+B)+(B+C)>(C+A)\ and\ analogs)$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}\text{ form sides of } a \text{ triangle}$$
$$\text{with } area\ F\ (say)\ and\ 16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2$$
$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$
$$\text{Now},\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}$$
$$=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}$$
$$\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$
$$\text{We have : LHS}=\frac{\sqrt[3]{ab}.\sqrt{a^3+b^3}\left(\frac{\sqrt[4]{b^5+c^5}}{\sqrt[3]{a}}+\frac{\sqrt[4]{c^5+a^5}}{\sqrt[3]{b}}\right)}{\sqrt[3]{bc}.\sqrt{b^3+c^3}+\sqrt[3]{ca}.\sqrt{c^3+a^3}}$$
$$+\frac{\sqrt[3]{bc}.\sqrt{b^3+c^3}\left(\frac{\sqrt[4]{a^5+b^5}}{\sqrt[3]{c}}+\frac{\sqrt[4]{c^5+a^5}}{\sqrt[3]{b}}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{ca}.\sqrt{c^3+a^3}}+\frac{\sqrt[3]{ca}.\sqrt{c^3+a^3}\left(\frac{\sqrt[4]{a^5+b^5}}{\sqrt[3]{c}}+\frac{\sqrt[4]{b^5+c^5}}{\sqrt[3]{a}}\right)}{\sqrt[3]{ab}.\sqrt{a^3+b^3}+\sqrt[3]{bc}.\sqrt{b^3+c^3}}$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(\begin{array}{c}x=\sqrt[3]{ab}.\sqrt{a^3+b^3},y=\sqrt[3]{bc}.\sqrt{b^3+c^3},z=\sqrt[3]{ca}.\sqrt{c^3+a^3},\\ A=\frac{\sqrt[4]{a^5+b^5}}{\sqrt[3]{c}},B=\frac{\sqrt[4]{b^5+c^5}}{\sqrt[3]{a}},C=\frac{\sqrt[4]{c^5+a^5}}{\sqrt[3]{b}}\end{array}\right)$$
$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$
$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{\sqrt[4]{(a^5+b^5)(b^5+c^5)}}{\sqrt[3]{ca}}}$$

$$\overset{A-G}{\geq}\sqrt{9\cdot\sqrt[3]{\frac{\sqrt{(a^5+b^5)(b^5+c^5)(c^5+a^5)}}{\sqrt[3]{a^2b^2c^2}}}}\overset{\substack{\text{Cesaro}\\ \text{and}\\ \because abc=1}}{\geq}3\cdot\sqrt{\sqrt[3]{\sqrt{8(abc)^5}}}\overset{abc=1}{=}3\cdot\sqrt[12]{2^3}$$

$$=3\cdot\sqrt[4]{2}\ \forall a,b,c>0\,\big|\,abc=1,''=''\ \text{iff } a=b=c=1\ (QED)$$

1516. If $a,b>0,\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=2$ then:

$$\frac{1}{a^2+b+2b\sqrt{a}}+\frac{1}{b^2+a+2a\sqrt{b}}+\frac{\lambda}{a+b}\leq\frac{\lambda+1}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=2\ or,1\overset{AM-GM}{\leq}\sqrt[4]{ab}\ or\ ab\geq1$$

$$\frac{1}{a^2+b+2b\sqrt{a}}+\frac{1}{b^2+a+2a\sqrt{b}}+\frac{\lambda}{a+b}\overset{AM-GM}{\leq}$$

$$\leq\frac{2}{2a\sqrt{b}+2b\sqrt{a}}+\frac{\lambda}{2\sqrt{ab}}=\frac{1}{(ab)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)}+\frac{\lambda}{2\sqrt{ab}}\leq\frac{\lambda+1}{2},$$

$$since\ ab\geq1\ and\ \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}=2$$

1517. If $a,b,c\geq0$ and $a+b+c+abc=4$, then prove that :

$$3(a^2+b^2+c^2)+13(ab+bc+ca)\geq48$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$4=\sum_{cyc}a+abc\overset{A-G}{\geq}3\cdot\sqrt[3]{abc}+abc\Rightarrow t^3+3t-4\leq0\ (t=\sqrt[3]{abc})$$

$$\Rightarrow(t-1)(t^2+t+4)\leq0\Rightarrow t=\sqrt[3]{abc}\leq1\Rightarrow abc\leq1\Rightarrow4-abc\geq3$$

$$\overset{a+b+c+abc=4}{\Leftrightarrow}\sum_{cyc}a\geq3\rightarrow(1)$$

$$\text{Now},3(a^2+b^2+c^2)+13(ab+bc+ca)=3\left(\sum_{cyc}a\right)^2+7\sum_{cyc}ab$$

$$\geq3\left(\sum_{cyc}a\right)^2+7\cdot\sqrt{3abc\sum_{cyc}a}\overset{a+b+c+abc=4}{=}$$

$$3\left(\sum_{cyc} a\right)^2+7.\sqrt{3\left(4-\sum_{cyc} a\right)\left(\sum_{cyc} a\right)}\overset{?}{\geq} 48\left(4-\sum_{cyc} a=abc\geq 0\right)$$

$$\Leftrightarrow 7.\sqrt{3x(4-x)}\overset{?}{\geq} 48-3x^2\left(x=\sum_{cyc} a\right)\Leftrightarrow 7.\sqrt{3x(4-x)}\overset{?}{\geq} 3(4-x)(4+x)$$

$$\Leftrightarrow 7.\sqrt{x}\overset{?}{\geq}\sqrt{3(4-x)}(4+x)\Leftrightarrow 49x\overset{?}{\geq} 3(4-x)(4+x)^2$$

$$\Leftrightarrow 3x^3+12x^2+x-192\overset{?}{\geq} 0\Leftrightarrow (x-3)(3x^2+21x+64)\overset{?}{\geq} 0\to \text{true}$$

$$\because x=\sum_{cyc} a\geq 3\ (\text{via } (1))\therefore 3(a^2+b^2+c^2)+13(ab+bc+ca)\geq 48$$

$$\forall\, a,b,c>0\,\middle|\, a+b+c+abc=4,''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1518. If $a,b>0$, then:

$$\frac{a^3+b^3}{2}\leq\left(\frac{a^2+b^2}{a+b}\right)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

Put $a=r\cos\theta, b=r\sin\theta, r>0, 0<\theta<\frac{\pi}{2}$

The given inequality can be written as

$$\frac{r^3}{2}(\sin^3\theta+\cos^3\theta)\leq\frac{r^6(\cos^2\theta+\sin^2\theta)^3}{r^3(\cos\theta+\sin\theta)^3}$$

$$\Leftrightarrow (\sin\theta+\cos\theta)^4(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)\leq 2$$

$$\Leftrightarrow (1+\sin 2\theta)^2\left(1-\frac{1}{2}\sin 2\theta\right)\leq 2\Leftrightarrow (1+2\sin 2\theta+\sin^2 2\theta)(2-\sin 2\theta)\leq 4$$

$$\Leftrightarrow 2+4\sin 2\theta+2\sin^2 2\theta-\sin 2\theta-2\sin^2 2\theta-\sin^3 2\theta\leq 4$$

$$\Leftrightarrow \sin^3 2\theta-3\sin 2\theta+2\geq 0\Leftrightarrow (1-\sin 2\theta)^2(2+\sin 2\theta)\geq 0$$

which is true. Equality when $\theta=\frac{\pi}{4}$ or when $a=b$.

Solution 2 by Ravi Prakash-New Delhi-India

For $a,b>0$, consider

$$2(a^2+b^2)^3-(a+b)^3(a^3+b^3)$$

$$=2(a^6+3a^4b^2+3a^2b^4+b^6)-(a^3+3a^2b+3ab^2+b^3)(a^3+b^3)$$

$$= 2a^6 + 6a^4b^2 + 6a^2b^4 + 2b^6 - \left(a^6 + 3a^5b + 3a^4b^2 + 2a^3b^3 + 3a^2b^4 + 3ab^5 + b^6\right)$$

$$= a^6 - 3a^5b + 3a^4b^2 - 2a^3b^3 - 3ab^5 + 3a^2b^4 + b^6$$

$$= (a-b)^4(a^2+ab+b^2) \geq 0$$

Equality when $a = b$. Thus:

$$\frac{a^3+b^3}{2} \leq \left(\frac{a^2+b^2}{a+b}\right)^3$$

Equality when $a = b$.

1519. If $a, b > 0$ and $a + b = ab$, then prove that :

$$\frac{1}{a^2+2a} + \frac{1}{b^2+2b} + \sqrt{(1+a^2)(1+b^2)} \geq \frac{21}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^2+2a} + \frac{1}{b^2+2b} + \sqrt{(1+a^2)(1+b^2)} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{4}{a^2+b^2+2(a+b)} + \sqrt{1+(a+b)^2-2ab+a^2b^2} \overset{a+b=ab}{=}$$

$$\frac{4}{a^2+b^2+2ab} + \sqrt{1+(a+b)^2-2(a+b)+(a+b)^2} = \frac{4}{x^2} + \sqrt{2x^2-2x+1}$$

$$\overset{?}{\geq} \frac{21}{4}\ (x = a+b) \Leftrightarrow \sqrt{2x^2-2x+1} \underset{(*)}{\overset{?}{\geq}} \frac{21x^2-16}{4x^2}$$

$$\because a+b = ab \therefore 4(a+b) \leq (a+b)^2 \Rightarrow x \geq 4 \to (1)$$

$$\therefore \frac{21x^2-16}{4x^2} = \frac{17x^2+4(x^2-16)}{4x^2} \geq \frac{17}{4} > 0 \therefore (*) \Leftrightarrow$$

$$(2x^2-2x+1) \overset{?}{\geq} \frac{(21x^2-16)^2}{16x^4} \Leftrightarrow 16x^4(2x^2-2x+1) \overset{?}{\geq} (21x^2-16)^2$$

$$\Leftrightarrow 32x^6 - 32x^5 - 425x^4 + 672x^2 - 256 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-4)\left((x-4)(32x^4+224x^3+855x^2+3256x+13040) + 52224\right) \overset{?}{\geq} 0$$

$$\to \text{true via } (1) \Rightarrow (*) \text{ is true } \therefore \frac{1}{a^2+2a} + \frac{1}{b^2+2b} + \sqrt{(1+a^2)(1+b^2)} \geq \frac{21}{4}$$

$$\forall\, a, b > 0 \mid a+b = ab, '' ='' \text{ iff } a = b = 2 \text{ (QED)}$$

1520. If $a, b, c > 0$ and $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 12$, then prove that :

$$\frac{1}{2a+3b+3c} + \frac{1}{2b+3c+3a} + \frac{1}{2c+3a+3b} \leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}=$$
$$\frac{1}{a+b+a+c+2(b+c)}+\frac{1}{b+c+b+a+2(c+a)}+\frac{1}{c+a+c+b+2(a+b)}$$
$$=\frac{1}{2x+y+z}+\frac{1}{2y+z+x}+\frac{1}{2z+x+y}\ (x=b+c,y=c+a,z=a+b)$$
$$=\frac{1}{x+y+z+x}+\frac{1}{y+z+x+y}+\frac{1}{z+x+y+z}=\frac{1}{Y+Z}+\frac{1}{Z+X}+\frac{1}{X+Y}\to(1)$$
$$(X=y+z,Y=z+x,Z=x+y)$$
Now, $X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0$
$\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form sides of a* triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding
$$2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\Rightarrow x=s-X,y=s-Y,z=s-Z$$
$$\therefore\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=12\Rightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=12$$
$$\Rightarrow\frac{1}{s-X}+\frac{1}{s-Y}+\frac{1}{s-Z}=12\Rightarrow\frac{\sum_{cyc}(s-Y)(s-Z)}{(s-X)(s-Y)(s-Z)}=12\Rightarrow\frac{4Rr+r^2}{r^2s}=12$$
$$\Rightarrow s=\frac{4R+r}{12r}\to(2)$$
$$\text{Now, via (1)},\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}$$
$$=\frac{\sum_{cyc}(Z+X)(X+Y)}{(X+Y)(Y+Z)(Z+X)}=\frac{\sum_{cyc}X^2+3\sum_{cyc}XY}{2s(s^2+2Rr+r^2)}=\frac{\left(\sum_{cyc}X\right)^2+\sum_{cyc}XY}{2s(s^2+2Rr+r^2)}$$
$$=\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\overset{\text{via (2)}}{=}\frac{5s^2+4Rr+r^2}{\left(\frac{4R+r}{6r}\right)(s^2+2Rr+r^2)}\overset{?}{\le}3$$
$$\Leftrightarrow(4R+r)(s^2+2Rr+r^2)-2r(5s^2+4Rr+r^2)\underset{(*)}{\overset{?}{\ge}}0$$
$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}$$
$$(4R+r)(16Rr-5r^2+2Rr+r^2)-2r(5(4R^2+4Rr+3r^2)+4Rr+r^2)$$
$$=2r(16R^2-23Rr-18r^2)=2r(R-2r)(16R+9r)\overset{\text{Euler}}{\ge}0\Rightarrow(*)\text{ is true}$$
$$\therefore\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}\le3$$
$$\forall a,b,c>0\,\Big|\,\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=12,''='' \text{ iff } a=b=c=\frac{1}{8}\ \text{(QED)}$$

1521.

If $x,y,z>0$ and $x(x+1)+y(y+1)+z(z+1)\le18$, then prove that :

$$\frac{1}{x+y+1}+\frac{1}{y+z+1}+\frac{1}{z+x+1}\ge\frac{3}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}}\ \sum_{cyc} x^2 \geq 12 \text{ and then}: \frac{1}{x+y+1}+\frac{1}{y+z+1}+\frac{1}{z+x+1} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{9}{2\sum_{cyc} x+3} \geq \frac{9}{36-2\sum_{cyc} x^2+3}\left(\because \sum_{cyc} x(x+1) \leq 18 \Rightarrow \sum_{cyc} x \leq 18-\sum_{cyc} x^2\right)$$

$$\overset{?}{\geq} \frac{3}{5} \Leftrightarrow \frac{3}{39-2\sum_{cyc} x^2} \overset{?}{\geq} \frac{1}{5} \Leftrightarrow 15 \overset{?}{\geq} 39-2\sum_{cyc} x^2$$

$$\left(\because -2\sum_{cyc} x^2 \geq 2\sum_{cyc} x-36 \Rightarrow 39-2\sum_{cyc} x^2 \geq 2\sum_{cyc} x+3>0\right)$$

$$\Leftrightarrow \sum_{cyc} x^2 \overset{?}{\geq} 12 \to \text{true} \therefore \sum_{cyc} \frac{1}{x+y+1} \geq \frac{3}{5}$$

$$\boxed{\text{Case 2}}\ \sum_{cyc} x^2 \leq 12 \text{ and then}: \frac{1}{x+y+1}+\frac{1}{y+z+1}+\frac{1}{z+x+1} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{9}{2\sum_{cyc} x+3} \overset{\text{CBS}}{\geq} \frac{9}{2\sqrt{3\sum_{cyc} x^2}+3} \overset{\sum_{cyc} x^2 \leq 12}{\geq} \frac{9}{2\sqrt{3.12}+3}=\frac{9}{15}$$

$$\therefore \sum_{cyc} \frac{1}{x+y+1} \geq \frac{3}{5} \text{ and so, combining both cases,}$$

$$\frac{1}{x+y+1}+\frac{1}{y+z+1}+\frac{1}{z+x+1} \geq \frac{3}{5}$$

$$\forall x,y,z>0 \mid x(x+1)+y(y+1)+z(z+1) \leq 18 \text{ (QED)}$$

1522. If $a,b,c>0, ab+bc+ca=3$ then:

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2} \geq \frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2} \geq \frac{1}{1+xy}\ \forall x,y \geq 0$$

Proof:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}=\frac{xy(x-y)^2+(xy-1)^2}{(1+x)^2(1+y)^2(1+xy)} \geq 0$$

Back to the problem:

$$LHS=\frac{1}{2}\sum\left[\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\right]\ge\frac{1}{2}\sum\frac{1}{1+ab}\overset{Bergstrom}{\ge}\frac{1}{2}\frac{9}{\sum ab+3}=\frac{3}{4}$$

1523. If $a,b,c>0$ and $a^2+b^2+c^2=12$, then prove that:

$$\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}}\ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{a^3+1}}\overset{?}{\ge}\frac{7-2a}{9}\Leftrightarrow\frac{1}{a^3+1}\overset{?}{\ge}\frac{(7-2a)^2}{81}$$

$$\left(\because a^2<12\Rightarrow a<2\sqrt{3}<\frac{7}{2}\Rightarrow 7-2a>0\right)\Leftrightarrow(a^3+1)(7-2a)^2-81\overset{?}{\le}0$$

$$\Leftrightarrow 4a^5-28a^4+49a^3+4a^2-28a-32\overset{?}{\le}0$$

$$\Leftrightarrow(a-2)^2(4a^3-12a^2-15a-8)\overset{?}{\le}0\Leftrightarrow 4a^3-12a^2-15a-8\overset{?}{<}0$$

$$\Leftrightarrow(a-4)(2a+1)^2-4\overset{?}{<}0\to\text{true}\because a<\frac{7}{2}<4\Rightarrow a-4<0$$

$$\therefore\frac{1}{\sqrt{a^3+1}}\ge\frac{7-2a}{9}\text{ and analogs}\Rightarrow\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}}\ge\sum_{cyc}\frac{7-2a}{9}$$

$$=\frac{7}{3}-\frac{2}{9}\sum_{cyc}a\overset{CBS}{\ge}\frac{7}{3}-\frac{2}{9}\cdot\sqrt{3\sum_{cyc}a^2}\overset{a^2+b^2+c^2=12}{=}\frac{7}{3}-\frac{2}{9}.6=1$$

$$\Rightarrow\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}}\ge 1$$

$$\forall a,b,c>0\,\middle|\,a^2+b^2+c^2=12,''=''\text{ iff } a=b=c=2\text{ (QED)}$$

1524. If $a,b,c>0$ and $ab+bc+ca=2abc$, then prove that:

$$\frac{a}{c(c+a)}+\frac{b}{a(a+b)}+\frac{c}{b(b+c)}\ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{c(c+a)}+\frac{b}{a(a+b)}+\frac{c}{b(b+c)}=\frac{a+c-c}{c(c+a)}+\frac{b+a-a}{a(a+b)}+\frac{c+b-b}{b(b+c)}=$$

$$\frac{1}{c}+\frac{1}{a}+\frac{1}{b}-\frac{1}{c+a}-\frac{1}{a+b}-\frac{1}{b+c}=\frac{ab+bc+ca}{abc}-\sum_{cyc}\frac{1}{b+c}=2-\sum_{cyc}\frac{1}{b+c}\ge 1$$

$$\overset{ab+bc+ca=2abc}{\Leftrightarrow}\sum_{cyc}\frac{1}{b+c}\overset{(*)}{\leq}\frac{ab+bc+ca}{2abc}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\Rightarrow a=s-x, b=s-y, c=s-z$

$$\therefore abc=r^2s\rightarrow(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(2)\therefore(*)\Leftrightarrow\sum_{cyc}\frac{1}{x}=\frac{s^2+4Rr+r^2}{4Rrs}\leq\frac{4Rr+r^2}{2r^2s}$$

$$\Leftrightarrow 2R(4R+r)\geq s^2+4Rr+r^2\Leftrightarrow s^2\leq 8R^2-2Rr-r^2\rightarrow\text{true}$$

$$\because s^2\overset{\text{Gerretsen}}{\leq}4R^2+4Rr+3r^2\overset{?}{\leq}8R^2-2Rr-r^2\Leftrightarrow 4R^2-6Rr-4r^2\overset{?}{\geq}0$$

$$\Leftrightarrow 2(R-2r)(2R+r)\overset{?}{\geq}0\rightarrow\text{true via Euler}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a}{c(c+a)}+\frac{b}{a(a+b)}+\frac{c}{b(b+c)}\geq 1$$

$$\forall a,b,c>0\,|\,ab+bc+ca=2abc,''=''\text{ iff }a=b=c=\frac{3}{2}\text{ (QED)}$$

1525. If $a,b,c>0$ and $ab+bc+ca+2abc=1$, then prove that :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)\geq\frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$1=2abc+\sum_{cyc}ab\overset{A-G}{\geq}2abc+3.\sqrt[3]{a^2b^2c^2}\Rightarrow 2x^3+3x^2-1\leq 0$$

$$(x=\sqrt[3]{abc})\Rightarrow(2x-1)(x+1)^2\leq 0\Rightarrow 2x\leq 1\Rightarrow\sqrt[3]{abc}\leq\frac{1}{2}\Rightarrow abc\leq\frac{1}{8}\rightarrow(1)$$

$$\text{Now, }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)-\frac{9}{2}=\frac{\sum_{cyc}ab-abc\sum_{cyc}a-\frac{9}{2}.abc}{abc}\geq$$

$$\frac{\sum_{cyc}ab-\frac{1}{3}\left(\sum_{cyc}ab\right)^2-\frac{9}{2}.abc}{abc}\overset{ab+bc+ca+2abc=1}{=}\frac{1-2t-\frac{(1-2t)^2}{3}-\frac{9t}{2}}{abc}\overset{?}{\geq}0\ (t=abc)$$

$$\Leftrightarrow 8t^2+31t-4\overset{?}{\leq}0\Leftrightarrow(8t-1)(t+4)\overset{?}{\leq}0\Leftrightarrow t=abc\overset{?}{\leq}\frac{1}{8}\rightarrow\text{true via }(1)$$

$$\therefore\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)\geq\frac{9}{2}$$

$$\forall a,b,c>0\,|\,ab+bc+ca+2abc=1,''=''\text{ iff }a=b=c=\frac{1}{2}\text{ (QED)}$$

1526. If $x, y > 0$ and $x + y \le 2$, then prove that :

$$x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right) \ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right)$$

$$= 2(x^2+y^2) - (x^3+y^3) + \frac{x+y}{xy} - xy(x+y) \overset{2 \ge x+y}{\ge}$$

$$(x+y)((x+y)^2 - 2xy) - \left((x+y)^3 - 3xy(x+y)\right) + \frac{x+y}{xy} - xy(x+y)$$

$$= (x+y)^3 - 2xy(x+y) - (x+y)^3 + 2xy(x+y) + \frac{x+y}{xy} \ge \frac{4(x+y)}{(x+y)^2} \overset{x+y \le 2}{\ge} \frac{4}{2}$$

$$\Rightarrow x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right) \ge 2$$

$$\forall\, x, y > 0 \,\middle|\, x + y \le 2, '' = '' \text{ iff } x = y = 1 \text{ (QED)}$$

1527. If $a, b, c > 0$ and $ab + bc + ca + abc = 4$ then prove that $a + b + c + abc \ge 4$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

We need to show

$$a + b + c + abc \ge 4 \text{ or}, a + b + c + abc \ge ab + bc + ca + abc \text{ or}$$

$$a + b + c \ge ab + bc + ca --- (1)$$

$$\text{Let } a = \frac{2x}{y+z}, b = \frac{2y}{z+x}, c = \frac{2z}{x+y}, \text{now from } (1) \sum \frac{x}{y+z} \ge \sum \frac{2xy}{(y+z)(z+x)}$$

$$\text{or}, \sum x(x+y)(x+z) \ge \sum 2xy(x+y) \text{ or}$$

$$\sum x^3 + 3xyz \ge \sum xy(x+y) \text{ (Schur's inequality).}$$

$$\text{Equality for } x = y = z = 1 \text{ or } a = b = c = 1$$

1528. **If $a,b,c>0$ and $a+b+c=abc$, then prove that:**

$$(a^2-1)(b^2-1)(c^2-1)\le\sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via }(1)\text{ and }(3)}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4),$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via }(1),(2)\text{ and }(3)}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\to(5)$$

$$\text{Now, }(a^2+1)(b^2+1)(c^2+1)\overset{a+b+c=abc}{=}$$

$$\left(a^2+\frac{abc}{\sum_{cyc}a}\right)\left(b^2+\frac{abc}{\sum_{cyc}a}\right)\left(c^2+\frac{abc}{\sum_{cyc}a}\right)$$

$$=\frac{abc}{\left(\sum_{cyc}a\right)^3}\left(a\sum_{cyc}a+bc\right)\left(b\sum_{cyc}a+ca\right)\left(c\sum_{cyc}a+ab\right)$$

$$\overset{a+b+c=abc}{=}\frac{abc}{a^3b^3c^3}\left(abc\left(\sum_{cyc}a\right)^3+a^2b^2c^2+abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2b^2\right)\right)$$

$$\overset{\text{via }(1),(2),(4)\text{ and }(5)}{=}\frac{r^2s^4+r^4s^2+r^2s^2(s^2-8Rr-2r^2)+r^2s^2\left((4R+r)^2-2s^2\right)}{r^4s^2}$$

$$=\frac{16R^2}{r^2}\Rightarrow\sqrt{(a^2+1)(b^2+1)(c^2+1)}=\frac{4R}{r}\to(\text{i})$$

$$\text{Again, } (a^2-1)(b^2-1)(c^2-1) = a^2b^2c^2 - 1 - \sum_{cyc} a^2b^2 + \sum_{cyc} a^2 \overset{a+b+c=abc}{=}$$

$$\left(\frac{\sum_{cyc} a}{abc}\right)^3 \left(a^2b^2c^2 - \left(\frac{abc}{\sum_{cyc} a}\right)^3 - \left(\sum_{cyc} a^2b^2\right)\left(\frac{abc}{\sum_{cyc} a}\right) + \left(\sum_{cyc} a^2\right)\left(\frac{abc}{\sum_{cyc} a}\right)^2\right)$$

$$\overset{\text{via (1),(2),(4) and (5)}}{=} \left(\frac{s}{r^2 s}\right)^3 \left(r^4s^2 - r^6 - r^4\left((4R+r)^2 - 2s^2\right) + r^4\left(s^2 - 8Rr - 2r^2\right)\right)$$

$$= \frac{4s^2 - 16R^2 - 16Rr - 4r^2}{r^2} \overset{\text{Gerretsen}}{\leq} \frac{4(4R^2+4Rr+3r^2) - 16R^2 - 16Rr - 4r^2}{r^2}$$

$$= 8 \overset{\text{Euler}}{\leq} \frac{4R}{r} \overset{\text{via (i)}}{=} \sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

$$\therefore (a^2-1)(b^2-1)(c^2-1) \leq \sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

$$\forall\, a,b,c > 0 \,\middle|\, a+b+c = abc, \text{"="} \text{ iff } a = b = c = \sqrt{3} \text{ (QED)}$$

1529. If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\left(\frac{x}{1+xy}\right)^2 + \left(\frac{y}{1+yz}\right)^2 + \left(\frac{z}{1+zx}\right)^2 \geq \frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{x}{1+xy}\right)^2 + \left(\frac{y}{1+yz}\right)^2 + \left(\frac{z}{1+zx}\right)^2 \overset{xyz=1}{=}$$

$$\left(\frac{x}{1+\frac{1}{z}}\right)^2 + \left(\frac{y}{1+\frac{1}{x}}\right)^2 + \left(\frac{z}{1+\frac{1}{y}}\right)^2 = \left(\frac{xz}{z+1}\right)^2 + \left(\frac{xy}{x+1}\right)^2 + \left(\frac{yz}{y+1}\right)^2$$

$$\overset{xyz=1}{=} \left(\frac{1}{yz+y}\right)^2 + \left(\frac{1}{zx+z}\right)^2 + \left(\frac{1}{xy+x}\right)^2$$

$$\geq \left(\frac{1}{xy+x}\right)\left(\frac{1}{yz+y}\right) + \left(\frac{1}{yz+y}\right)\left(\frac{1}{zx+z}\right) + \left(\frac{1}{zx+z}\right)\left(\frac{1}{xy+x}\right)$$

$$= \frac{zx+z+xy+x+yz+y}{(xy+x)(yz+y)(zx+z)} \overset{xyz=1}{=} \frac{\sum_{cyc} xy + \sum_{cyc} x}{1 + xyz + \sum_{cyc} xy + \sum_{cyc} x} \overset{?}{\geq} \frac{3}{4}$$

$$\Leftrightarrow \sum_{cyc} xy + \sum_{cyc} x \overset{?}{\geq} 3 + 3xyz \overset{xyz=1}{=} 6 \to \text{true} \because$$

$$\sum_{cyc} xy + \sum_{cyc} x \overset{A-G}{\geq} 3.\sqrt[3]{x^2y^2z^2} + 3.\sqrt[3]{xyz} \overset{xyz=1}{=} 6$$

$$\therefore \left(\frac{x}{1+xy}\right)^2 + \left(\frac{y}{1+yz}\right)^2 + \left(\frac{z}{1+zx}\right)^2 \geq \frac{3}{4}$$

$$\forall\, x,y,z > 0 \,\middle|\, xyz = 1, \text{"="} \text{ iff } x = y = z = 1 \text{ (QED)}$$

1530. If $a,b,c>0$ and $a^2+b^2+c^2=\frac{3}{4}$, then prove that :

$$\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\le -1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}s^2-2(4Rr+r^2)$$

$$\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$$

$$\text{Now, }\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)+1=\frac{(a-1)(b-1)(c-1)+abc}{abc}$$

$$=\frac{\sum_{cyc}a-\sum_{cyc}ab+2abc-1}{abc}\le 0\Leftrightarrow\sum_{cyc}ab+1\ge\sum_{cyc}a+2abc$$

$$\overset{a^2+b^2+c^2=\frac{3}{4}}{\Leftrightarrow}\sum_{cyc}ab+\frac{4}{3}\sum_{cyc}a^2\ge\left(\sum_{cyc}a\right).\sqrt{\frac{4}{3}\sum_{cyc}a^2}+\frac{2abc}{\sqrt{\frac{4}{3}\sum_{cyc}a^2}}$$

$$=\frac{\frac{4}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+2abc}{\sqrt{\frac{4}{3}\sum_{cyc}a^2}}\Leftrightarrow$$

$$\left(\frac{4}{3}\sum_{cyc}a^2\right)\left(\sum_{cyc}ab+\frac{4}{3}\sum_{cyc}a^2\right)^2\ge\left(\frac{4}{3}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+2abc\right)^2$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\left(4\sum_{cyc}a^2+3\sum_{cyc}ab\right)^2\ge 3\left(2\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+3abc\right)^2$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}(s^2-8Rr-2r^2)\left(4(s^2-8Rr-2r^2)+3(4Rr+r^2)\right)^2$$

$$\ge 3\left(2s(s^2-8Rr-2r^2)+3r^2s\right)^2$$

$$\Leftrightarrow 2s^6-(48Rr+30r^2)s^4+r^2(456R^2+372Rr+51r^2)s^2$$

$$-25r^3(4R+r)^3 \overset{(*)}{\geq} 0 \text{ and } \because 2(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to}$$
$$\text{prove } (*), \text{it suffices to prove : LHS of } (*) \geq 2(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (48R-60r)s^4 - r(1080R^2-1332Rr+99r^2)s^2$$
$$+r^2(6592R^3-8880R^2r+2100Rr^2-275r^3) \overset{(**)}{\geq} 0$$
$$\text{and} \because (48R-60r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (**), \text{it suffices to prove : LHS of } (**) \geq (48R-60r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (456R^2-1068Rr+501r^2)s^2 \overset{(***)}{\geq} r\begin{pmatrix}5696R^3-14160R^2r+8700Rr^2\\-1225r^3\end{pmatrix}$$
$$\text{Now}, 456R^2-1068Rr+501r^2 = (R-2r)(456R-156r)+189r^2 \overset{\text{Euler}}{\geq} 189r^2$$
$$> 0 \therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq} (456R^2-1068Rr+501r^2)(16Rr-5r^2)$$
$$\overset{?}{\geq} r(5696R^3-14160R^2r+8700Rr^2-1225r^3)$$
$$\Leftrightarrow 200t^3-651t^2+582t-160 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(74t^2+126t(t-2)+t+80) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)+1 \leq 0$$
$$\Rightarrow \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) \leq -1$$
$$\forall a,b,c>0 \mid a^2+b^2+c^2=\frac{3}{4}, ''='' \text{ iff } a=b=c=\frac{1}{2} \text{ (QED)}$$

1531. If $a,b,c \in \mathbb{R}$ and $a+b+c=3$, then prove that :

$$\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6} \geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}} \text{ Exactly two among } a,b,c \text{ equal to zero : WLOG we may assume}$$
$$b=c=0\ (a=3) \text{ and then : LHS} = \sqrt[3]{24}+2.\sqrt[3]{6} \approx 6.51874 > 6$$
$$\boxed{\text{Case 2}} \text{ Exactly one among } a,b,c \text{ equals zero : WLOG we may assume } a=0$$
$$(b,c>0 \text{ with } b+c=3) \text{ and then : LHS} = \sqrt[3]{6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6} \overset{?}{>} 6$$
$$\Leftrightarrow \left(\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\right)^3 \underset{(*)}{\overset{?}{\geq}} (6-\sqrt[3]{6})^3$$
$$\text{Now, LHS of } (*) = 2b^2+6+2c^2+6$$
$$+3.\sqrt[3]{(2b^2+6)(2c^2+6)}.\left(\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\right) \overset{\text{Jensen}}{\geq}$$
$$(b+c)^2+12+3.\sqrt[3]{4b^2c^2+6(b+c)^2+36}.2.\sqrt[3]{2\left(\frac{b+c}{2}\right)^2+6}$$

$$\left(\because f(x)=\sqrt[3]{2x^2+6}\ \forall\, x\in(0,3)\text{ is convex as } f''(x)=\frac{8(9-x^2)}{9(2x^2+6)^{\frac{5}{3}}}>0\right)$$

$$\overset{b+c=3}{=}21+6.\sqrt[3]{4b^2c^2+90.\sqrt[3]{\frac{21}{2}}}>21+6.\sqrt[3]{45.21}\approx 79.87919$$

$$>\left(6-\sqrt[3]{6}\right)^3(\approx 73.185667)\therefore\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}>6$$

Case 3 $a,b,c>0$ and then : $\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\overset{\text{Jensen}}{\geq}$

$$3.\sqrt[3]{2\left(\frac{a+b+c}{3}\right)^2+6}=6\therefore\text{combining all cases,}$$

$$\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\geq 6$$

$$\forall\, a,b,c\in\mathbb{R}\,\big|\,a+b+c=3,''=''\text{ iff } a=b=c=1\text{ (QED)}$$

1532. If $a,b,c,d\in[1,2]$, then prove that :

$$(a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)\leq 25$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Via Schweitzer (1914), for $0<m<M$ and $x_i\in[m,M]$ $(i\in\overline{1,n})$,

$$\left(\frac{1}{n}\sum_{cyc}x_i\right)\left(\frac{1}{n}\sum_{cyc}\frac{1}{x_i}\right)\leq\frac{(m+M)^2}{4mM}\to(1)$$

Choosing $n=4, x_1=a^2, x_2=b^2, x_3=c^2, x_4=d^2$ and $\because a,b,c,d\in[1,2]$

$$\therefore a^2,b^2,c^2,d^2\in[1,4]\Rightarrow m=1, M=4\text{ and so, via }(1),$$

$$\left(\frac{a^2+b^2+c^2+d^2}{4}\right)\left(\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}{4}\right)\leq\frac{(1+4)^2}{4.1.4}$$

$$\Rightarrow(a^2+b^2+c^2+d^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)\leq 25\ \forall\, a,b,c,d\in[1,2]\text{ (QED)}$$

1533. If $a,b,c\in\mathbb{R}$ and $ab+bc+ca=3$, then prove that :

$$(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)\geq 64$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ Exactly one among a,b,c equals zero : WLOG we may assume $a=0$ $(b,c>0$ with $bc=3)$ and then :

$$(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)=(b^2+c^2)(2b^2+c^2)(2c^2+b^2)$$
$$=(b^2+c^2)\left(2(b^2+c^2)^2+b^2c^2\right)\overset{A-G}{\geq}2bc(9b^2c^2)=18.27>64$$
$$\Rightarrow(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)>64$$

$\boxed{Case\ 2}$ $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say); so $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y,$ $c=s-z\therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{via\ (1)\ and\ (3)}{=}$$
$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4),$$
$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{via\ (1),(2)\ and\ (3)}{=}(4Rr+r^2)^2-2r^2s.s$$
$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\to(5)$$

Now, $(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)\geq 64\overset{\because ab+bc+ca=3}{\Leftrightarrow}$

$$\left(a^2+\sum_{cyc}a^2\right)\left(b^2+\sum_{cyc}a^2\right)\left(c^2+\sum_{cyc}a^2\right)\geq\frac{64}{27}\left(\sum_{cyc}ab\right)^3$$
$$\Leftrightarrow 27a^2b^2c^2+54\left(\sum_{cyc}a^2\right)^3+27\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)\geq 64\left(\sum_{cyc}ab\right)^3$$
$$\overset{via\ (2),(3),(4)\ and\ (5)}{\Leftrightarrow}27r^4s^2+54(s^2-8Rr-2r^2)^3$$
$$+27r^2(s^2-8Rr-2r^2)\left((4R+r)^2-2s^2\right)-64(4Rr+r^2)^3\ \boxed{\overset{(*)}{\geq}}\ 0\text{ and}$$
$\because 54(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\geq}0\therefore$ in order to prove $(*)$, it suffices to prove :
$$\text{LHS of }(*)\geq 54(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow(324R-297r)s^4-r(7668R^2-7938Rr+810r^2)s^2$$
$$+r^2(46496R^3-58440R^2r+14550Rr^2-1825r^3)\ \boxed{\overset{(**)}{\geq}}\ 0$$
and $\because(324R-297r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(**)\geq(324R-297r)(s^2-16Rr+5r^2)^2\Leftrightarrow$

$$\left(1350R^2-2403Rr+1080r^2\right)s^2\ \boxed{\overset{(***)}{\geq}}\ r\left(\begin{matrix}18224R^3-34716R^2r+20535Rr^2\\-2800r^3\end{matrix}\right)$$

$$\text{Again, }\left(1350R^2-2403Rr+1080r^2\right)s^2\overset{\text{Gerretsen}}{\geq}$$

$$\left(1350R^2-2403Rr+1080r^2\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq} r\left(18224R^3-34716R^2r+20535Rr^2-2800r^3\right)$$

$$\Leftrightarrow 1688t^3-5241t^2+4380t-1300\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(1688t+1511)+3672\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\Rightarrow(**)$$

$$\Rightarrow(*)\text{ is true}\therefore\left(2a^2+b^2+c^2\right)\left(2b^2+c^2+a^2\right)\left(2c^2+a^2+b^2\right)\geq 64\text{ and so,}$$

$$\text{combining both cases, }\left(2a^2+b^2+c^2\right)\left(2b^2+c^2+a^2\right)\left(2c^2+a^2+b^2\right)\geq 64$$

$$\forall a,b,c\in\mathbb{R}\,\middle|\,ab+bc+ca=3,''=''\text{ iff }a=b=c=1\ (QED)$$

1534.

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{\sqrt[5]{c}\left(b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{a}+a.\sqrt[5]{b}}+\frac{\sqrt[5]{a}\left(b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}\right)}{b.\sqrt[5]{c}+c.\sqrt[5]{b}}+\frac{\sqrt[5]{b}\left(a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}\right)}{a.\sqrt[5]{c}+c.\sqrt[5]{a}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A,B,C>0,(A+B),(B+C),(C+A)\text{ form sides of a triangle}$$

$$(\because (A+B)+(B+C)>(C+A)\text{ and analogs})$$

$$\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}\text{ form sides of a triangle with area F (say)}$$

$$\text{and }16F^2=2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2$$

$$=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

We have : $\frac{\sqrt[5]{c}(b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b})}{b.\sqrt[5]{a}+a.\sqrt[5]{b}}+\frac{\sqrt[5]{a}(b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b})}{b.\sqrt[5]{c}+c.\sqrt[5]{b}}+\frac{\sqrt[5]{b}(a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a})}{a.\sqrt[5]{c}+c.\sqrt[5]{a}}$

$$=\frac{\frac{\sqrt[5]{a}}{a}\left(a.\frac{b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}}{bc}\right)}{\frac{b.\sqrt[5]{c}+c.\sqrt[5]{b}}{bc}}+\frac{\frac{\sqrt[5]{b}}{b}\left(b.\frac{a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}}{ca}\right)}{\frac{a.\sqrt[5]{c}+c.\sqrt[5]{a}}{ca}}+\frac{\frac{\sqrt[5]{c}}{c}\left(c.\frac{b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}}{ab}\right)}{\frac{b.\sqrt[5]{a}+a.\sqrt[5]{b}}{ab}}$$

$$\overset{abc=1}{=}\frac{\frac{\sqrt[5]{a}}{a}\left(\frac{b^2.\sqrt[3]{c}+c^2.\sqrt[3]{b}}{b^2c^2}\right)}{\frac{\sqrt[5]{b}}{b}+\frac{\sqrt[5]{c}}{c}}+\frac{\frac{\sqrt[5]{b}}{b}\left(\frac{a^2.\sqrt[3]{c}+c^2.\sqrt[3]{a}}{c^2a^2}\right)}{\frac{\sqrt[5]{c}}{c}+\frac{\sqrt[5]{a}}{a}}+\frac{\frac{\sqrt[5]{c}}{c}\left(\frac{b^2.\sqrt[3]{a}+a^2.\sqrt[3]{b}}{a^2b^2}\right)}{\frac{\sqrt[5]{a}}{a}+\frac{\sqrt[5]{b}}{b}}$$

$$=\frac{\frac{\sqrt[5]{a}}{a}\left(\frac{\sqrt[3]{b}}{b^2}+\frac{\sqrt[3]{c}}{c^2}\right)}{\frac{\sqrt[5]{b}}{b}+\frac{\sqrt[5]{c}}{c}}+\frac{\frac{\sqrt[5]{b}}{b}\left(\frac{\sqrt[3]{c}}{c^2}+\frac{\sqrt[3]{a}}{a^2}\right)}{\frac{\sqrt[5]{c}}{c}+\frac{\sqrt[5]{a}}{a}}+\frac{\frac{\sqrt[5]{c}}{c}\left(\frac{\sqrt[3]{a}}{a^2}+\frac{\sqrt[3]{b}}{b^2}\right)}{\frac{\sqrt[5]{a}}{a}+\frac{\sqrt[5]{b}}{b}}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{\sqrt[5]{a}}{a},y=\frac{\sqrt[5]{b}}{b},z=\frac{\sqrt[5]{c}}{c},A=\frac{\sqrt[3]{a}}{a^2},B=\frac{\sqrt[3]{b}}{b^2},C=\frac{\sqrt[3]{c}}{c^2}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt[3]{a}}{a^2}.\frac{\sqrt[3]{b}}{b^2}\right)}\overset{A-G}{\geq}$$

$$\sqrt{9.\sqrt[3]{\frac{\sqrt[3]{a^2b^2c^2}}{a^4b^4c^4}}}\overset{abc=1}{=}\sqrt{9}=3\ \forall\, a,b,c>0\,|\,abc=1,''=''\ \text{iff } a=b=c=1\ \text{(QED)}$$

1535. If $a,b,c>0$ then:

$$a^{1-\frac{1}{a}}+b^{1-\frac{1}{b}}+c^{1-\frac{1}{c}}\geq 27^{\frac{1}{a+b+c}}\cdot(a+b+c)^{\frac{a+b+c-3}{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\frac{a^{1-\frac{1}{a}}+b^{1-\frac{1}{b}}+c^{1-\frac{1}{c}}}{a+b+c}=\frac{a\left(a^{-\frac{1}{a}}\right)+b\left(b^{-\frac{1}{b}}\right)+c\left(c^{-\frac{1}{c}}\right)}{a+b+c}\geq$$

$$\geq\left[\left(a^{-\frac{1}{a}}\right)^a\left(b^{-\frac{1}{b}}\right)^b\left(c^{-\frac{1}{c}}\right)^c\right]^{\frac{1}{(a+b+c)}}=\frac{1}{(abc)^{\frac{1}{(a+b+c)}}}\qquad (1)$$

Also, $(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3} \Rightarrow \frac{1}{abc} \geq \frac{27}{(a+b+c)^3}$ (2)

Thus, from (1) and (2)

$$a^{1-\frac{1}{a}} + b^{1-\frac{1}{b}} + c^{1-\frac{1}{c}} \geq (a+b+c)\left(\frac{27}{(a+b+c)^3}\right)^{\frac{1}{a+b+c}}$$

$$= 27^{\frac{1}{a+b+c}} \cdot (a+b+c)^{1-\frac{3}{a+b+c}} = 27^{\frac{1}{a+b+c}} \cdot (a+b+c)^{\frac{a+b+c-3}{a+b+c}}$$

Equality holds for $a = b = c$.

1536. If $a, b, c > 0$ then:

$$a \cdot 2^{\frac{1}{a}} + b \cdot 2^{\frac{1}{b}} + c \cdot 2^{\frac{1}{c}} \geq (a+b+c) \cdot 8^{\frac{1}{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Using $AM \geq GM$

$$\frac{a\left(2^{\frac{1}{a}}\right) + b\left(2^{\frac{1}{b}}\right) + c\left(2^{\frac{1}{c}}\right)}{a+b+c} \geq \left[\left(2^{\frac{1}{a}}\right)^a \left(2^{\frac{1}{b}}\right)^b \left(2^{\frac{1}{c}}\right)^c\right]^{\frac{1}{a+b+c}}$$

$$\Rightarrow a\left(2^{\frac{1}{a}}\right) + b\left(2^{\frac{1}{b}}\right) + c\left(2^{\frac{1}{c}}\right) \geq (a+b+c)(8)^{\frac{1}{a+b+c}}$$

Equality holds for $a = b = c$.

Solution 2 by Khaled Abd Imouti-Syria

$$f(x) = x \cdot 2^{\frac{1}{x}} = x \cdot e^{\frac{1}{x}\ln(2)}, \quad f'(x) = e^{\frac{1}{x}\ln(2)} + \left(-\frac{x}{x^2}\ln(2)\right) \cdot e^{\frac{1}{x}\ln(2)}$$

$$f'(x) = \left(1 - \frac{1}{x}\ln(2)\right) \cdot e^{\frac{1}{x}\ln(2)}$$

$$f''(x) = \frac{1}{x^2}\ln(2) \cdot e^{\frac{1}{x}\ln(2)} - \frac{1}{x^2}\ln(2)\, e^{\frac{1}{x}\ln(2)} \cdot \left(1 - \frac{1}{x}\ln(2)\right)$$

$$f''(x) = \frac{1}{x^2}\ln(2) \cdot e^{\frac{1}{x}\ln(2)} \cdot \left[1 - 1 + \frac{1}{x}\ln(2)\right] = \frac{1}{x^3 \ln^2(2)\, e^{\frac{1}{x}\ln(2)}}$$

$f''(x) = \frac{1}{x^2}\ln^2(2) \cdot 2^{\frac{1}{x}} > 0$. So f is convex function and then:

$$a\cdot 2^{\frac{1}{a}}+b\cdot 2^{\frac{1}{b}}+c\cdot 2^{\frac{1}{c}}\ge 3\cdot\left(\frac{a+b+c}{3}\right)\cdot 2^{\frac{1}{\left(\frac{a+b+c}{3}\right)}}$$

$$a\cdot 2^{\frac{1}{a}}+b\cdot 2^{\frac{1}{b}}+c\cdot 2^{\frac{1}{c}}\ge (a+b+c)\cdot(8)^{\frac{1}{a+b+c}}$$

Equality holds for $a=b=c$.

1537. If $a,b,c>0$ then:

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}\ge\sqrt{2}(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}$$

$$=a|c+ib|+b|a+ic|+c|b+ia|=|ac+iab|+|ba+ibc|+|bc+iac|$$

$$\ge|ac+ba+bc+i(ab+bc+ac)|=(ab+bc+ca)|1+i|$$

$$=\sqrt{2}(ab+bc+ca)$$

Equality holds for: $a=b=c$.

Solution 2 by Tapas Das-India

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}$$

$$=\sqrt{a^2c^2+a^2b^2}+\sqrt{b^2a^2+b^2c^2}+\sqrt{c^2b^2+c^2a^2}$$

$$\overset{Minkowski}{\ge}\sqrt{(ac+ba+cb)^2+(ab+bc+ca)^2}$$

$$=\sqrt{2(ab+bc+ca)^2}=\sqrt{2}(ab+bc+ca)$$

Equality holds for: $a=b=c$.

Solution 3 by Khaled Abd Imouti-Syria

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}\ge\sqrt{2}(ab+bc+ca)$$

$$\sqrt{a^2c^2+a^2b^2}+\sqrt{b^2a^2+c^2b^2}+\sqrt{c^2b^2+c^2a^2}\overset{?}{\ge}\sqrt{2}(ab+bc+ca)$$

$$\underbrace{\sqrt{\frac{a^2c^2+a^2b^2}{2}}+\sqrt{\frac{b^2a^2+c^2b^2}{2}}+\sqrt{\frac{c^2b^2+c^2a^2}{2}}}_{e_1}\ge ab+bc+ca$$

By using AM-GM

$$e_1 \geq \frac{ac+ab+bc+cb+cb+ca}{2} = \frac{2(ac+ab+bc)}{2}$$

$$e_1 \geq ac+ab+bc$$

Equality holds for: $a=b=c$.

Solution 4 by Kunihiko Chikaya-Tokyo-Japan

$$\sqrt{\frac{b^2+c^2}{2}} \overset{QM-AM}{\geq} \frac{b+c}{2} \Rightarrow a\sqrt{\frac{b^2+c^2}{2}} \overset{QM-AM}{\geq} \frac{a(b+c)}{2} \Rightarrow$$

$$\Rightarrow a\sqrt{b^2+c^2} \geq \frac{1}{\sqrt{2}}(ab+ac)$$

$$\sum_{cyc} a\sqrt{b^2+c^2} \geq \frac{1}{\sqrt{2}}\sum_{cyc}(ab+ac) = \frac{2}{\sqrt{2}}\sum_{cyc} ab = \sqrt{2}(ab+bc+ca)$$

Equality holds for: $a=b=c$.

1538. If $x,y,z>0, x+y+z=1$ then:

$$\frac{1}{x}\left(\frac{y^3}{x+z}+\frac{z^3}{x+y}\right)+\frac{1}{y}\left(\frac{x^3}{y+z}+\frac{z^3}{y+x}\right)+\frac{1}{z}\left(\frac{x^3}{z+y}+\frac{y^3}{z+x}\right) \geq 1$$

Proposed by Lamiye Quliyeva-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

Let's group the expression on the left in another form:

$$\underbrace{\frac{1}{x}\cdot\frac{y^3}{x+z}}_{II}+\underbrace{\frac{1}{x}\cdot\frac{z^3}{x+y}}_{III}+\underbrace{\frac{1}{y}\cdot\frac{x^3}{y+z}}_{I}+\underbrace{\frac{1}{y}\cdot\frac{z^3}{x+y}}_{III}+\underbrace{\frac{1}{z}\cdot\frac{x^3}{y+z}}_{I}+\underbrace{\frac{1}{z}\cdot\frac{y^3}{x+z}}_{II} \geq 1$$

$$\underbrace{\left(\frac{1}{y}\cdot\frac{x^3}{y+z}+\frac{1}{z}\cdot\frac{x^3}{y+z}\right)}_{I}+\underbrace{\left(\frac{1}{x}\cdot\frac{y^3}{x+z}+\frac{1}{z}\cdot\frac{y^3}{x+z}\right)}_{II}+\underbrace{\left(\frac{1}{x}\cdot\frac{z^3}{x+y}+\frac{1}{y}\cdot\frac{x^3}{y+z}\right)}_{III} \geq 1$$

$$\frac{x^3}{y+z}\cdot\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{y^3}{x+z}\cdot\left(\frac{1}{x}+\frac{1}{z}\right)+\frac{z^3}{x+y}\cdot\left(\frac{1}{x}+\frac{1}{y}\right) \geq 1$$

$$\frac{x^3}{y+z}\cdot\frac{y+z}{yz}+\frac{y^3}{x+z}\cdot\frac{x+z}{xz}+\frac{z^3}{x+y}\cdot\frac{x+y}{xy} \geq 1$$

$$\frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy} \geq 1 \quad (Both\ side\ mutliply\ xyz>0)$$

From here

$$x^4+y^4+z^4 \geq xyz\ (*)$$

Because:

$$x^4+y^4+z^4=\left(\frac{1}{2}x^4+\frac{1}{2}y^4\right)+\left(\frac{1}{2}x^4+\frac{1}{2}z^4\right)+\left(\frac{1}{2}y^4+\frac{1}{2}z^4\right)\quad (1)$$

$$\left.\begin{array}{l}\frac{1}{2}x^4+\frac{1}{2}y^4\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^4\frac{1}{2}y^4}=x^2y^2\\ \frac{1}{2}x^4+\frac{1}{2}z^4\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^4\frac{1}{2}z^4}=x^2z^2\\ \frac{1}{2}y^4+\frac{1}{2}z^4\overset{A-G}{\geq}2\sqrt{\frac{1}{2}y^4\frac{1}{2}z^4}=y^2z^2\end{array}\right\}(2)$$

(2) using in (1)

$$x^4+y^4+z^4\geq x^2y^2+x^2z^2+y^2z^2=\left(\frac{1}{2}x^2y^2+\frac{1}{2}y^2z^2\right)+\left(\frac{1}{2}y^2z^2+\frac{1}{2}x^2z^2\right)+\left(\frac{1}{2}x^2y^2+\frac{1}{2}x^2z^2\right)\quad (3)$$

$$\left.\begin{array}{l}\frac{1}{2}x^2y^2+\frac{1}{2}y^2z^2\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^2y^2\frac{1}{2}y^2z^2}=xy^2z\\ \frac{1}{2}y^2z^2+\frac{1}{2}x^2z^2\overset{A-G}{\geq}2\sqrt{\frac{1}{2}y^2z^2\frac{1}{2}x^2z^2}=xyz^2\\ \frac{1}{2}x^2y^2+\frac{1}{2}x^2z^2\overset{A-G}{\geq}2\sqrt{\frac{1}{2}x^2y^2\frac{1}{2}x^2z^2}=x^2yz\end{array}\right\}(4)$$

(4) using in (3)

$$x^4+y^4+z^4\geq xy^2z+xyz^2+x^2yz=xyz(x+y+z)=xyz$$

So, $x^4+y^4+z^4\geq xyz\ (*)$

Equality holds if $x=y=z=\frac{1}{3}$

1539. If $a,b,c\geq-\frac{3}{2}$ and $abc+ab+bc+ca+a+b+c\geq 0$, then prove that : $a+b+c\geq 0$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $a+\frac{3}{2}=x, b+\frac{3}{2}=y, c+\frac{3}{2}=z$ and then : $a=\frac{2x-3}{2}, b=\frac{2y-3}{2}$,

$$c=\frac{2z-3}{2}\Rightarrow abc+ab+bc+ca+a+b+c\geq 0$$

$$\Leftrightarrow \frac{1}{8}\prod_{cyc}(2x-3)+\frac{1}{4}\sum_{cyc}\big((2x-3)(2y-3)\big)+\frac{1}{2}\sum_{cyc}(2x-3)\geq 0$$

$$\Leftrightarrow 2\sum_{cyc}x+8xyz\geq 4\sum_{cyc}xy+9\to(1)\text{ and we are to prove }:\frac{1}{2}\sum_{cyc}(2x-3)\geq 0$$

$$\Leftrightarrow 2\sum_{cyc}x\geq 9\to(*)$$

$$\boxed{\text{Case 1}}\ 2xyz\geq\sum_{cyc}xy\text{ and }\because x,y,z\geq 0\therefore 2xyz\geq\sum_{cyc}xy\overset{A-G}{\geq}3\sqrt[3]{x^2y^2z^2}$$

$$\Rightarrow 8x^3y^3z^3\geq 27x^2y^2z^2\Rightarrow xyz\geq\frac{27}{8}\Rightarrow\sqrt[3]{xyz}\geq\frac{3}{2}\therefore\sum_{cyc}x\overset{A-G}{\geq}3\sqrt[3]{xyz}\geq\frac{9}{2}$$

$$\Rightarrow(*)\text{ is true}$$

$$\boxed{\text{Case 2}}\ \sum_{cyc}xy\geq 2xyz\text{ and then, via }(1),2\sum_{cyc}x-9\geq 4\left(\sum_{cyc}xy-2xyz\right)\geq 0$$

$$\Rightarrow(*)\text{ is true}\therefore\text{ combining both cases},(*)\text{ is true }\forall\,x,y,z\geq 0\text{ constrained by }(1)$$

$$\therefore a+b+c\geq 0\ \forall\,a,b,c\geq -\frac{3}{2}\,\Big|\,abc+ab+bc+ca+a+b+c\geq 0\ (\text{QED})$$

1540. If $x,y,z\in[-1,1]$ and $x+y+z+xyz=0$, then prove that:

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x+y+z+xyz=0\Rightarrow x(1+yz)=-y-z\Rightarrow x=-\frac{y+z}{1+yz}$$

$$\Rightarrow x+1=1-\frac{y+z}{1+yz}=\frac{1-y-z(1-y)}{1+yz}\Rightarrow x+1=\frac{(1-y)(1-z)}{1+yz}\to(1)$$

$$\boxed{\text{Case 1}}\ xyz\geq 0\text{ and then}:\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\overset{CBS}{\leq}\sqrt{3}.\sqrt{\sum_{cyc}x+3}$$

$$\overset{x+y+z+xyz=0}{=}\sqrt{3}.\sqrt{3-xyz}\overset{xyz\geq 0}{\leq}\sqrt{3}.\sqrt{3}\therefore\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$$

$\boxed{\text{Case 2}}\ xyz\leq 0$ and then: either $\boxed{\text{Case 2i}}\ x,y,z\leq 0$ so that: $\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq\sqrt{1}+\sqrt{1}+\sqrt{1}\therefore\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$

or $\boxed{\text{Case 2ii}}$ one variable ≤ 0 and the other two ≥ 0 and WLOG we may assume $x\leq 0;y,z\geq 0$ and $\because m^2+4m+4\geq 4m+4\ \forall\,m\geq 0$

$$\therefore m+1\leq\frac{(m+2)^2}{4}\Rightarrow\sqrt{m+1}\leq 1+\frac{m}{2}\text{ and so},\sqrt{y+1}+\sqrt{z+1}\leq 1+\frac{y}{2}+1+\frac{z}{2}$$

$$\Rightarrow \sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \le \sqrt{\frac{(1-y)(1-z)}{1+yz}}+2+\frac{y+z}{2}$$

$$\left(\text{note}: \frac{(1-y)(1-z)}{1+yz} \ge 0 \because 0 \le y,z \le 1\right)$$

$$\Rightarrow \sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}-3 \le \sqrt{\frac{(1-y)(1-z)}{1+yz}}+\frac{y+z}{2}-1$$

$$= \sqrt{\frac{(1-y)(1-z)}{1+yz}}-\frac{(1-y)+(1-z)}{2} \overset{A-G}{\le}$$

$$\sqrt{\frac{(1-y)(1-z)}{1+yz}}-\sqrt{(1-y)(1-z)} \begin{pmatrix} \because (1-y),(1-z) \ge 0 \\ as\ 0 \le y,z \le 1 \end{pmatrix}$$

$$= \sqrt{(1-y)(1-z)}.\left(\frac{1}{\sqrt{1+yz}}-1\right) = \sqrt{(1-y)(1-z)}.\frac{\frac{1}{1+yz}-1}{\frac{1}{\sqrt{1+yz}}+1}$$

$$= -\sqrt{(1-y)(1-z)}.\frac{yz}{(1+yz)\left(\frac{1}{\sqrt{1+yz}}+1\right)} \le 0 \because 0 \le y,z \le 1$$

$$\therefore \sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \le 3 \therefore \text{combining } all \text{ cases},$$

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \le 3\ \forall\ x,y,z \in [-1,1] \,\middle|\, x+y+z+xyz=0,$$

$$'' ='' \text{ iff } x=y=z=0 \text{ (QED)}$$

1541.

If $a,b,c \in \mathbb{R}$ and $ab(a+b)+bc(b+c)+ca(c+a) \ge 6$,

then prove that : $a^2+b^2+c^2 \ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$6 \le \sum_{cyc} bc(b+c) \overset{CBS}{\le} \sqrt{\sum_{cyc} a^2b^2}.\sqrt{\sum_{cyc}(b+c)^2}$$

$$\le \sqrt{\frac{\left(\sum_{cyc} a^2\right)^2}{3}}.\sqrt{2\sum_{cyc} a^2+2\sum_{cyc} ab} \le \sqrt{\frac{\left(\sum_{cyc} a^2\right)^2}{3}}.\sqrt{4\sum_{cyc} a^2} \Rightarrow \frac{4\left(\sum_{cyc} a^2\right)^3}{3} \ge 36$$

$$\Rightarrow \sum_{cyc} a^2 \geq 3 \text{ (QED)}$$

1542. If $a, b, c \geq 0$ and $a+b+c=2$, then prove that :

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} \leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables equal zero and WLOG we may assume $b=c=0$ $(a=2)$ and then : $\text{LHS} = \sqrt{2^2} = 2 < 3$

$\boxed{\text{Case 2}}$ Exactly 1 variable equals zero and WLOG we may assume $a=0$ $(b+c=2$ with $b, c>0)$ and then : $\text{LHS} = \sqrt{bc}+b+c \overset{A-G}{\leq} \dfrac{3(b+c)}{2} = 3$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and assigning $b+c=x, c+a=y, a+b=z \Rightarrow$ $x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z$, $y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say); so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s$ $\rightarrow (1) \Rightarrow a = s-x, b=s-y, c=s-z \therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3)$,

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4), \text{ and } \sum_{cyc} a^2b^2 =$$

$$\left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R+r)^2 - 2s^2\right) \rightarrow (5)$$

$$\text{Now, } \left(\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\right)^2 - 9 \overset{a+b+c=2}{=}$$

$$\sum_{cyc} a^2 + \sum_{cyc} ab + 2\sum_{cyc}\sqrt{(b^2+ca)(c^2+ab)} - \frac{9}{4}\left(\sum_{cyc} a\right)^2$$

$$\overset{CBS}{\leq}\sum_{cyc}a^2+\sum_{cyc}ab+2\sqrt{3}.\sqrt{\sum_{cyc}\left((b^2+ca)(c^2+ab)\right)}-\frac{9}{4}\left(\sum_{cyc}a\right)^2\overset{?}{<}0$$

$$\Leftrightarrow 2\sqrt{3}.\sqrt{\sum_{cyc}a^2b^2+\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-c^2\right)\right)+abc\sum_{cyc}a}\overset{?}{<}\frac{5\sum_{cyc}a^2+14\sum_{cyc}ab}{4}$$

$$\Leftrightarrow 2\sqrt{3}.\sqrt{\sum_{cyc}a^2b^2+\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)}\overset{?}{<}\frac{5\sum_{cyc}a^2+14\sum_{cyc}ab}{4}$$

$$\Leftrightarrow\left(5\sum_{cyc}a^2+14\sum_{cyc}ab\right)^2\overset{?}{>}192\left(\sum_{cyc}a^2b^2+\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)\right)$$

$$\overset{\text{via }(3),(4)\text{ and }(5)}{\Leftrightarrow}\left(5(s^2-8Rr-2r^2)+14(4Rr+r^2)\right)^2$$

$$\overset{?}{>}192\left(r^2((4R+r)^2-2s^2)+(s^2-8Rr-2r^2)(4Rr+r^2)\right)$$

$$\Leftrightarrow 25s^4-(608Rr-232r^2)s^2+r^2(3328R^2+1664Rr+208r^2)\underset{(*)}{\overset{?}{>}}0\text{ and}$$

$$\because 25(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$\text{LHS of }(*)>25(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(64R-6r)s^2\overset{(**)}{>}r(1024R^2-1888Rr+139r^2)$$

$$\text{Now, }(64R-6r)s^2\overset{Gerretsen}{\geq}(64R-6r)(16Rr-5r^2)\overset{?}{>}$$

$$r(1024R^2-1888Rr+139r^2)\Leftrightarrow r^2(1472R-109r)\overset{?}{>}0\rightarrow\text{true}$$

$$\because 1472R-109r\overset{Euler}{\geq}2944r-1097r>0\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\right)^2-9<0$$

$$\Rightarrow\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}<3\ \forall\ a,b,c>0\,|\,a+b+c=2$$

$$\therefore\text{ combining } all \text{ cases, }\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}\leq 3$$

$$\forall\ a,b,c\geq 0\,|\,a+b+c=2,''=''\ \text{iff }(a=0,b=c=1)$$

$$\text{or }(b=0,c=a=1)\text{ or }(c=0,a=b=1)\text{ (QED)}$$

1543. If $a,b\in\mathbb{R}$ and $2(a^2+b^2)=3ab+1$, then prove that :

$$a^3+b^3\leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } a+b\le 0, \text{then}: a^3+b^3=(a+b)\left(\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2\right)\le 0<2$$

$$\text{and so, we now focus on}: a+b>0 \text{ and then}: a^3+b^3\le 2 \Leftrightarrow (a^3+b^3)^2\le 4$$

$$\Leftrightarrow (a+b)^2(a^2+b^2-ab)^2\le 4 \Leftrightarrow (a^2+b^2+2ab)(a^2+b^2-ab)^2\le 4$$

$$\overset{\because 2(a^2+b^2)=3ab+1}{\Leftrightarrow}\left(\frac{3ab+1}{2}+2ab\right)\left(\frac{3ab+1}{2}-ab\right)^2\le 4 \Leftrightarrow (7x+1)(x+1)^2\le 32$$

$$(x=ab)\Leftrightarrow 7x^3+15x^2+9x-31\le 0 \Leftrightarrow (x-1)(7x^2+22x+31)\le 0$$

$$\Leftrightarrow x\le 1\left(\begin{matrix}\because \text{discriminant of } 7x^2+22x+31=22^2-4.7.31=-384<0\\ \Rightarrow 7x^2+22x+31>0\end{matrix}\right)$$

$$\to \text{true} \because 3ab+1=2(a^2+b^2)\ge 4ab \Rightarrow x=ab\le 1 \therefore a^3+b^3\le 2$$

$$\forall a,b\in\mathbb{R}\,\middle|\,2(a^2+b^2)=3ab+1, ''='' \text{ iff } a=b=1 \text{ (QED)}$$

1544. If $a,b,c\in\mathbb{R}$ and $a(1+b^2)+b(1+c^2)+c(1+a^2)\ge 6$, then prove that : $a^2+b^2+c^2\ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let us assume}: \sum_{cyc} a^2<3$$

$$\text{Now, } 6\le \sum_{cyc} a+\sum_{cyc} ab^2 \overset{CBS}{\le} \sum_{cyc} a+\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} \overset{assumption}{<}$$

$$\sum_{cyc} a+\sqrt{3\sum_{cyc} a^2b^2} \overset{\left(\sum_{cyc} a^2\right)^2\ge 3\sum_{cyc} a^2b^2}{\le} \sum_{cyc} a+\sum_{cyc} a^2 \overset{assumption}{<} \sum_{cyc} a+3$$

$$\Rightarrow \sum_{cyc} a>3 \to (*)$$

$$\text{But}: 3>\sum_{cyc} a^2\ge \frac{1}{3}\left(\sum_{cyc} a\right)^2 \Rightarrow \left(\sum_{cyc} a\right)^2-9<0$$

$$\Rightarrow\left(\sum_{cyc} a+3\right)\left(\sum_{cyc} a-3\right)<0\Rightarrow -3<\sum_{cyc} a<3\rightarrow(**)$$

$\therefore (*),(**)$ **contradict each other $\therefore$ our assumption must be incorrect and so,**

we conclude : $a^2+b^2+c^2\ge 3$ (QED)

1545. **If $a,b,c>0$ and $abc=1$, then prove that**

$$\frac{(ab)^{2024}(b^{2024}c^2+a^{2026})}{c^2(a^{2025}+b^{2025})}+\frac{(bc)^{2024}(c^{2024}a^2+b^{2026})}{a^2(b^{2025}+c^{2025})}+\frac{(ca)^{2024}(a^{2024}b^2+c^{2026})}{b^2(c^{2025}+a^{2025})}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x=a^{2024}b^{2026},\ \ y=b^{2024}c^{2026},\ \ z=c^{2024}a^{2026}$. **We have**

$$\sum_{cyc}\frac{(ab)^{2024}(b^{2024}c^2+a^{2026})}{c^2(a^{2025}+b^{2025})}=\sum_{cyc}\frac{(ab)^{2025}.c^{2024}(b^{2024}c^2+a^{2026})}{c^{2025}(a^{2025}+b^{2025})}=$$

$$=\sum_{cyc}\frac{(ab)^{2025}.(y+z)}{(bc)^{2025}+(ca)^{2025}}=\sum_{cyc}(ab)^{2025}\cdot\sum_{cyc}\frac{y+z}{(bc)^{2025}+(ca)^{2025}}-2\sum_{cyc}x\ge$$

$$\overset{CBS}{\ge}\sum_{cyc}(ab)^{2025}\cdot\frac{\left(\sum_{cyc}\sqrt{y+z}\right)^2}{2\sum_{cyc}(ab)^{2025}}-2\sum_{cyc}x=\sum_{cyc}\left(\sqrt{(x+y)(x+z)}-x\right)\ge$$

$$\overset{CBS}{\ge}\sum_{cyc}(x+\sqrt{yz}-x)=\sum_{cyc}\sqrt{yz}\overset{AM-GM}{\ge}3\sqrt[3]{xyz}=3(abc)^{1350}=3,$$

as desired. Equality holds iff $a=b=c=1$.

1546. **If $a,b,c>0$ then:**

$$\sum_{cyc}\frac{c^4.\sqrt{bc}(c+a)^2+b^4.\sqrt{bc}(a+b)^2}{c^2.\sqrt{ca}+b^2.\sqrt{ab}}\ge\frac{4}{3}.\sqrt{\frac{abc(a+b+c)}{3}}(a+b+c)^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A,B,C>0,(A+B),(B+C),(C+A)\text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C)>(C+A)\text{ and analogs})$$
$$\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}\text{ form sides of a triangle with area F (say) and }16F^2=$$
$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$
$$\text{Now},\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$
$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$
$$\text{We have}:\frac{c^4.\sqrt{bc}(c+a)^2+b^4.\sqrt{bc}(a+b)^2}{c^2.\sqrt{ca}+b^2.\sqrt{ab}}+\frac{c^4.\sqrt{ca}(b+c)^2+a^4.\sqrt{ca}(a+b)^2}{c^2.\sqrt{bc}+a^2.\sqrt{ab}}$$
$$+\frac{b^4.\sqrt{ab}(b+c)^2+a^4.\sqrt{ab}(c+a)^2}{b^2.\sqrt{bc}+a^2.\sqrt{ca}}$$
$$=\frac{\frac{c^2}{b^2}.\sqrt{bc}(c+a)^2+\frac{b^2}{c^2}.\sqrt{bc}(a+b)^2}{\frac{\sqrt{ca}}{b^2}+\frac{\sqrt{ab}}{c^2}}+\frac{\frac{c^2}{a^2}.\sqrt{ca}(b+c)^2+\frac{a^2}{c^2}.\sqrt{ca}(a+b)^2}{\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ab}}{c^2}}$$
$$+\frac{\frac{b^2}{a^2}.\sqrt{ab}(b+c)^2+\frac{a^2}{b^2}.\sqrt{ab}(c+a)^2}{\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ca}}{b^2}}$$
$$=\frac{\frac{\sqrt{bc}}{a^2}\left(\frac{c^2a^2(c+a)^2}{b^2}+\frac{a^2b^2(a+b)^2}{c^2}\right)}{\frac{\sqrt{ca}}{b^2}+\frac{\sqrt{ab}}{c^2}}+\frac{\frac{\sqrt{ca}}{b^2}\left(\frac{a^2b^2(a+b)^2}{c^2}+\frac{b^2c^2(b+c)^2}{a^2}\right)}{\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ab}}{c^2}}$$
$$+\frac{\frac{\sqrt{ab}}{c^2}\left(\frac{b^2c^2(b+c)^2}{a^2}+\frac{c^2a^2(c+a)^2}{b^2}\right)}{\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ca}}{b^2}}=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(\begin{array}{c}x=\frac{\sqrt{bc}}{a^2},y=\frac{\sqrt{ca}}{b^2},z=\frac{\sqrt{ab}}{c^2},\\ A=\frac{b^2c^2(b+c)^2}{a^2},B=\frac{c^2a^2(c+a)^2}{b^2},C=\frac{a^2b^2(a+b)^2}{c^2}\end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}$$

$$=\sqrt{3\sum_{cyc}\left(\frac{b^2c^2(b+c)^2}{a^2}.\frac{c^2a^2(c+a)^2}{b^2}\right)}=\sqrt{3\sum_{cyc}(c^4(c+a)^2(b+c)^2)}$$

$$\geq\sum_{cyc}\left(c^2(c+a)(b+c)\right)=\sum_{cyc}a^4+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)$$

$$\geq\frac{1}{3}\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)=\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)$$

$$\geq\frac{1}{9}\left(\sum_{cyc}a\right)^2\left(4\sum_{cyc}ab\right)\geq\frac{4}{9}\left(\sum_{cyc}a\right)^2\left(\sqrt{3abc\left(\sum_{cyc}a\right)}\right)$$

$$=\frac{4}{3}\left(\sum_{cyc}a\right)^2\left(\sqrt{\frac{3abc(\sum_{cyc}a)}{9}}\right)=\frac{4}{3}.\sqrt{\frac{abc(a+b+c)}{3}}(a+b+c)^2$$

$$\forall\, a,b,c>0,''=''\text{ iff } a=b=c\ (QED)$$

1547. If $a,b,c>0$ then:

$$\frac{1}{a^4(a+b)}+\frac{1}{b^4(b+c)}+\frac{1}{c^4(c+a)}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{a^4(a+b)}+\frac{1}{b^4(b+c)}+\frac{1}{c^4(c+a)}\overset{abc=1}{=}\sum\frac{a^4b^4c^4}{a^4(a+b)}=$$

$$=\sum\frac{(b^2c^2)^2}{a+b}\geq\frac{(\sum b^2c^2)^2}{2(a+b+c)}=\frac{(\sum b^2c^2)(\sum b^2c^2)}{2(a+b+c)}\overset{AM-GM}{\geq}$$

$$\geq\frac{3(abc)^{\frac{2}{3}}abc(a+b+c)}{2(a+b+c)}=\frac{3}{2}\ (since\ abc=1)$$

Equality holds for $a=b=c=1$.

1548. If $a,b,c>0$ and $a^2+b^2+c^2=1$, then prove that :

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{3\sqrt{3}+9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\sum_{cyc}\frac{1+a}{1-a^2}\overset{a^2+b^2+c^2=1}{=}\sum_{cyc}\frac{1+a}{b^2+c^2}$$

$$=\sum_{cyc}\frac{1}{b^2+c^2}+\sum_{cyc}\frac{a}{b^2+c^2}\overset{\text{Bergstrom}}{\geq}\frac{9}{2\sum_{cyc}a^2}+\sum_{cyc}\frac{a}{b^2+c^2}\overset{a^2+b^2+c^2=1}{=}$$

$$\frac{9}{2}+\sum_{cyc}\frac{a}{b^2+c^2}\overset{?}{\geq}\frac{3\sqrt{3}+9}{2}\Leftrightarrow\sum_{cyc}\frac{a}{b^2+c^2}\boxed{\overset{?}{\geq}}_{(*)}\frac{3\sqrt{3}}{2}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\rightarrow(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4),\text{ and}$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via (1),(2) and (3)}}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\rightarrow(5)$$

$$\text{Now, }\sum_{cyc}\frac{a}{b^2+c^2}=\sum_{cyc}\frac{a^4}{a^3b^2+a^3c^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}(a^3b^2+a^2b^3)}$$

$$=\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a-c\right)\right)}=\frac{\left(\sum_{cyc}a^2\right)^2}{\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)-abc\sum_{cyc}ab}\overset{?}{\geq}\frac{3\sqrt{3}}{2}\overset{a^2+b^2+c^2=1}{=}$$

$$\frac{3\sqrt{3}}{2\sqrt{\sum_{cyc}a^2}} \Leftrightarrow 4\left(\sum_{cyc}a^2\right)^5 \geq 27\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)-abc\sum_{cyc}ab\right)^2$$

$$\overset{via\ (1),(2),(3),(4)\ and\ (5)}{\Leftrightarrow}$$

$$4(s^2-8Rr-2r^2)^5-27\left(sr^2((4R+r)^2-2s^2)-r^2s(4Rr+r^2)\right)^2 \boxed{\overset{(**)}{\geq}} 0 \text{ and } \because$$

$$P = 4(s^2-16Rr+5r^2)^5+4r(40R-35r)(s^2-16Rr+5r^2)^4$$
$$+4r^2(640R^2-1120Rr+463r^2)(s^2-16Rr+5r^2)^3$$
$$+4r^3(5120R^3-13008R^2r+10572Rr^2-3025r^3)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\left(\begin{array}{c}\because 5120t^3-13008t^2+10572t-3025\ \left(t=\frac{R}{r}\right)\\ =(t-2)(5120t^2-2768t+5036)+7047 \overset{\text{Euler}}{\geq} 7047>0\end{array}\right)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \geq P \Leftrightarrow$

$$(4688R^4-14680R^3r+18093R^2r^2-10750Rr^3+2495r^4)s^2 \boxed{\overset{(***)}{\geq}}$$
$$r(73728R^5-248832R^4r+337536R^3r^2-232020R^2r^3+77085Rr^4-9117r^5)$$

$$\because 4688t^4-14680t^3+18093t^2-10750t+2495$$
$$=(t-2)(4688t^3-5304t^2+7485t+4220)+10935 \overset{\text{Euler}}{\geq} 10935>0$$

$$\therefore \text{LHS of } (***) \overset{\text{Rouche}}{\geq}$$
$$\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right)\left(\begin{array}{c}2R^2+10Rr-r^2\\-2(R-2r).\sqrt{R^2-2Rr}\end{array}\right)$$
$$\overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 9376R^6-56208R^5r+133530R^4r^2-163426R^3r^3$$
$$+111417R^2r^4-41385Rr^5+6622r^6$$
$$\overset{?}{\geq} 2(R-2r).\sqrt{R^2-2Rr}.\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right) \Leftrightarrow$$

$$(R-2r)(9376R^5-37456R^4r+58618R^3r^2-46190R^2r^3+19037Rr^4-3311r^5)$$
$$\boxed{\overset{?}{\underset{(****)}{\geq}}}\ 2(R-2r).\sqrt{R^2-2Rr}.\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right)$$

$$\text{and } \because 9376t^5-37456t^4+58618t^3-46190t^2+19037t-3311r^5$$
$$=(t-2)(24t^4+9352t^3(t-2)+21210t^2-3770t+11497)+19683 \overset{\text{Euler}}{\geq}$$

$19683>0$ and $\because R-2r \overset{\text{Euler}}{\geq} 0 \therefore$ in order to prove $(****)$, it suffices to prove :

$$(9376R^5-37456R^4r+58618R^3r^2-46190R^2r^3+19037Rr^4-3311r^5)^2$$
$$>4(R^2-2Rr)(4688R^4-14680R^3r+18093R^2r^2-10750Rr^3+2495r^4)^2$$
$$\Leftrightarrow 24002560t^9-139530240t^8+351802368t^7-468261888t^6+276650496t^5$$
$$+91963584t^4-285463896t^3+214237449t^2-76262814t+10962721 \boxed{>} 0$$
$$\Leftrightarrow (t-2)((t-2)Q+2554105446)+387420489>0, \text{ where } Q=$$
$$2242560t^7+21760000(t-2)+81712128t^5+32666624t^4+80468480t^3$$

$$+283171008t^2+525346216t+1182938281\to \text{true} \because t\overset{\text{Euler}}{\geq} 2\Rightarrow (****)$$

$$\Rightarrow (***)\Rightarrow (**)\Rightarrow (*)\text{ is true} \therefore \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq \frac{3\sqrt{3}+9}{2},$$

$$\forall a,b,c>0\,\Big|\,a^2+b^2+c^2=1,''=''\text{ iff } a=b=c=\frac{1}{\sqrt{3}}\text{ (QED)}$$

1550. **If $a,b,c\in\mathbb{R}$ and $(a+b+c)(a^2+1)(b^2+1)(c^2+1)=24$, then prove that : $a^2+b^2+c^2\geq 3$**

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$24=(a+b+c)(a^2+1)(b^2+1)(c^2+1)\leq \sqrt{3\sum_{cyc}a^2}.(a^2+1)(b^2+1)(c^2+1)=$$

$$=\sqrt{3\sum_{cyc}a^2}.\left(a^2b^2c^2+1+\sum_{cyc}a^2+\sum_{cyc}a^2b^2\right)$$

$$\leq\sqrt{3\sum_{cyc}a^2}.\left(\frac{1}{27}\left(\sum_{cyc}a^2\right)^3+1+\sum_{cyc}a^2+\frac{1}{3}\left(\sum_{cyc}a^2\right)^2\right)$$

$$=\sqrt{3\sum_{cyc}a^2}.\frac{\left(\sum_{cyc}a^2\right)^3+27+27\sum_{cyc}a^2+9\left(\sum_{cyc}a^2\right)^2}{27}$$

$$\Rightarrow 576\leq\frac{\left(3\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^3+27+27\sum_{cyc}a^2+9\left(\sum_{cyc}a^2\right)^2\right)^2}{729}$$

$$\Rightarrow t\left(t^3+9t^2+27t+27\right)^2-243.576\geq 0\ \left(t=\sum_{cyc}a^2\right)$$

$$\Rightarrow t^7+18t^6+135t^5+540t^4+1215t^3+1458t^2+729t-139968\geq 0$$

$$\Rightarrow (t-3)\left(t^6+21t^5+198t^4+1134t^3+4617t^2+15309t+46656\right)\geq 0$$

$$\Rightarrow t\geq 3\ \left(\because t\sum_{cyc}a^2\geq 0\right)\therefore a^2+b^2+c^2\geq 3$$

$$\forall a,b,c\in\mathbb{R}\,\Big|\,(a+b+c)(a^2+1)(b^2+1)(c^2+1)=24,$$

$$''=''\text{ iff } a=b=c=1\text{ (QED)}$$

1550. **If $x,y,z>0, x+y+z=1$ then:**

$$\sqrt{x+yz}+\sqrt{y+xz}+\sqrt{z+xy}\leq 2$$

Proposed by Shirvan Tahirov, Lamiye Quliyeva-Azerbaijan

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sqrt{x+yz}+\sqrt{y+xz}+\sqrt{z+xy}=$$

$$=\sqrt{(1-y-z)+zy}+\sqrt{(1-x-z)+xz}+\sqrt{(1-y-x)+xy}=$$

$$=\sqrt{(1-y)(1-z)}+\sqrt{(1-x)(1-z)}+\sqrt{(1-x)(1-y)}\le$$

$$\overset{AM-GM}{\le}\frac{(1-y)+(1-z)}{2}+\frac{(1-x)+(1-z)}{2}+\frac{(1-x)+(1-y)}{2}=$$

$$=\frac{6-2(x+y+z)}{2}=2$$

Equality holds for $x=y=z=\frac{1}{3}$.

1551. If $x,y,z>0, x+y+z=5$ then:

$$\sqrt[7]{5x+yz}+\sqrt[7]{5y+xz}+\sqrt[7]{5z+xy}<5$$

Proposed by Samed Ahmedov-Azerbaijan

Solution by Tapas Das-India

$$5x+yz=(x+y+z)x+yz=(x+y)(x+z)$$

$$\sqrt[7]{5x+yz}=\sqrt[7]{(x+y)(x+z)1\cdot 1\cdot 1\cdot 1\cdot 1}\overset{AM-GM}{<}$$

$$<\frac{x+y+x+z+5}{7}=\frac{2x+y+z+5}{7}$$

$$\sqrt[7]{5x+yz}+\sqrt[7]{5y+xz}+\sqrt[7]{5z+xy}=\sum_{cyc}\sqrt[7]{5x+yz}<$$

$$<\sum_{cyc}\frac{2x+y+z+5}{7}=\frac{2\cdot 5+5+5+15}{7}=\frac{35}{7}=5$$

1552. If $a,\mathrm{b}\in\mathbb{R}$ and $a\mathrm{b}(a^2+1)(\mathrm{b}^2+1)=4$, then prove that :

$$a^2+\mathrm{b}^2\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$ab(a^2+1)(b^2+1)=4\Rightarrow ab=\frac{4}{(a^2+1)(b^2+1)}>0\Rightarrow ab>0\rightarrow(1)$$

$$\text{Now, } ab(a^2+1)(b^2+1)=4\Rightarrow ab(a^2b^2+1+a^2+b^2)=4\Rightarrow a^2+b^2$$

$$=\frac{4}{ab}-a^2b^2-1\overset{?}{\geq}2\Leftrightarrow 4-x^3-3x\overset{?}{\geq}0\ \big(\because x=ab>0\ldots\text{via }(1)\big)$$

$$\Leftrightarrow x^3+3x-4\overset{?}{\leq}0\Leftrightarrow(x-1)(x^2+x+4)\overset{?}{\leq}0\Leftrightarrow x\underbrace{\overset{?}{\leq}}_{(*)}1\ (\because x^2+x+4>0)$$

$$\text{We have : } ab(a^2b^2+1+a^2+b^2)=4\Rightarrow 4=a^3b^3+ab+ab(a^2+b^2)$$

$$\overset{A-G}{\geq}a^3b^3+ab+ab(2ab)\ \big(\because ab>0\ldots\text{via }(1)\big)\Rightarrow x^3+2x^2+x-4\leq 0$$

$$\Rightarrow(x-1)(x^2+3x+4)\leq 0\Rightarrow x\leq 1\ (\because x^2+3x+4>0)\Rightarrow(*)\text{ is true}$$

$$\therefore a^2+b^2\geq 2\ \forall\ a,b\in\mathbb{R}\ \big|\ ab(a^2+1)(b^2+1)=4,$$

$$''=''\ \text{ iff }(a=b=1)\text{ or }(a=b=-1)\ \text{(QED)}$$

1553. If $a,b,c>0$ and $a^2+b^2+c^2+a+b+c\geq 6$, then prove that :

$$\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\geq\frac{3}{\sqrt{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$6\leq a^2+b^2+c^2+a+b+c\leq\sum_{cyc}a^2+\sqrt{3\sum_{cyc}a^2}=\frac{x^2}{3}+x$$

$$\left(x=\sqrt{3\sum_{cyc}a^2}\right)\Rightarrow x^2+3x-18\geq 0\Rightarrow(x+6)(x-3)\geq 0\Rightarrow x\geq 3\rightarrow(1)$$

$$\text{Now, WLOG assuming } a\geq b\geq c\Rightarrow a^2\geq b^2\geq c^2 \text{ and } \frac{1}{\sqrt{b+c}}\geq\frac{1}{\sqrt{c+a}}\geq\frac{1}{\sqrt{a+b}}$$

$$\therefore\text{ via Chebyshev, }\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\geq\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{\sqrt{b+c}}\right)$$

$$\overset{\text{Bergstrom}}{\geq}\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\frac{9}{\sum_{cyc}\sqrt{b+c}}\right)\overset{\text{CBS}}{\geq}\left(3\sum_{cyc}a^2\right)\left(\frac{1}{\sqrt{3}.\sqrt{2\sum_{cyc}a}}\right)$$

$$= \left(3\sum_{cyc} a^2\right)\left(\frac{1}{\sqrt{6}.\sqrt{\sqrt{3\sum_{cyc} a^2}}}\right) = \frac{x^2}{\sqrt{6x}} = \frac{x\sqrt{x}}{\sqrt{6}} \overset{via\,(1)}{\geq} \frac{3.\sqrt{3}}{\sqrt{3}.\sqrt{2}}$$

$$\Rightarrow \frac{a^2}{\sqrt{b+c}} + \frac{b^2}{\sqrt{c+a}} + \frac{c^2}{\sqrt{a+b}} \geq \frac{3}{\sqrt{2}}$$

$$\forall\, a,b,c > 0 \,\middle|\, a^2+b^2+c^2+a+b+c \geq 6 \text{ (QED)}$$

1554. If $a,b,c \in \mathbb{R}$ and $(a^3+1)(b^3+1)(c^3+1) = 729$,

then prove that : $a^2+b^2+c^2 \geq 12$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+2)^4 - 16(a^3+1)^2 = a^8 - 8a^6 + 24a^4 - 32a^3 + 32a^2$$

$$= a^2(a-2)^2\left(a^2(a+2)^2+8\right) \geq 0\ \forall\, a \in \mathbb{R} \therefore (a^3+1)^2 \overset{(1)}{\leq} \frac{(a^2+2)^4}{16}$$

$$\text{Now, } 729^2 = \prod_{cyc}(a^3+1)^2 \overset{via\,(1)\text{ and analogs}}{\leq} \prod_{cyc}\frac{(a^2+2)^4}{16}$$

$$\Rightarrow 27 \leq \frac{1}{8}\prod_{cyc}(a^2+2)\ \left(\because \prod_{cyc}(a^2+2) > 0\ \forall\, a,b,c \in \mathbb{R}\right)$$

$$\Rightarrow 216 \leq a^2b^2c^2 + 8 + 4\sum_{cyc} a^2 + 2\sum_{cyc} a^2b^2$$

$$\leq \frac{1}{27}\left(\sum_{cyc} a^2\right)^3 + 8 + 4\sum_{cyc} a^2 + \frac{2}{3}\left(\sum_{cyc} a^2\right)^2$$

$$\left(\because \left(\sum_{cyc} a^2\right)^3 \geq 27a^2b^2c^2 \text{ and } \left(\sum_{cyc} a^2\right)^2 \geq 3\sum_{cyc} a^2b^2\ \forall\, a,b,c \in \mathbb{R}\right)$$

$$= \frac{t^3 + 216 + 108t + 18t^2}{27}\left(t = \sum_{cyc} a^2\right) \Rightarrow t^3 + 18t^2 + 108t - 5616 \geq 0$$

$$\Rightarrow (t-12)(t^2+30t+48) \geq 0 \Rightarrow t \geq 12$$

$$\left(\because t^2+30t+48=\left(\sum_{cyc}a^2\right)^2+30\left(\sum_{cyc}a^2\right)+48>0\ \forall\, a,b,c\in\mathbb{R}\right)$$

$$\therefore a^2+b^2+c^2\ge 12\ \forall\, a,b,c\in\mathbb{R}\,\middle|\,(a^3+1)(b^3+1)(c^3+1)=729,$$

$$''=''\ \text{iff } a=b=c=2\ (\text{QED})$$

1555. **If $a,b,c>0$, then:**

$$(a+b+c)\left(\frac{1}{2a+3b}+\frac{1}{2b+3c}+\frac{1}{2c+3a}\right)+\frac{\sqrt[4]{(1+a)(1+b)(1+c)(1+\sqrt[3]{abc})}}{\sqrt[3]{(1+\sqrt{ab})(1+\sqrt{bc})(1+\sqrt{ca})}}\ge\frac{14}{5}$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$(a+b+c)\left(\frac{1}{2a+3b}+\frac{1}{2b+3c}+\frac{1}{2c+3b}\right)\ge\frac{(a+b+c).3^2}{(2a+3b)+(2b+3c)+(2c+3b)}=\frac{9}{5}.$$

So it suffices to prove that

$$\sqrt[4]{(1+a)(1+b)(1+c)(1+\sqrt[3]{abc})}\ge\sqrt[3]{(1+\sqrt{ab})(1+\sqrt{bc})(1+\sqrt{ca})}$$

$$\Leftrightarrow\ 3\left(\sum_{cyc}\ln(1+a)+\ln(1+\sqrt[3]{abc})\right)\ge 4\sum_{cyc}\ln(1+\sqrt{ab})\ \ (1)$$

Let $a=e^x, b=e^y, c=e^z,\ \ x,y,z\in\mathbb{R}$, and let $f(t)=\ln(1+e^t), t\in\mathbb{R}$. We have

$$(1)\Leftrightarrow 3\big(f(x)+f(y)+f(z)\big)+3f\left(\frac{x+y+z}{3}\right)\ge 4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right].$$

$f''(t)=\dfrac{e^t}{(1+e^t)^2}>0$, so f is convex, and by Popoviciu and Jensen inequalities, we get

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)\ge 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

$$f(x)+f(y)\ge 2f\left(\frac{x+y}{2}\right),\ \ f(y)+f(z)\ge 2f\left(\frac{y+z}{2}\right),\ \ f(z)+f(x)\ge 2f\left(\frac{z+x}{2}\right).$$

Adding these inequalities yields the desired inequality (1). So the proof is complete.

Equality holds iff $a=b=c$.

1556. If $a,b>0$ then:

$$\frac{a+b}{\sqrt{a(4a+5b)}+\sqrt{b(4b+5a)}}\ge\frac{1}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{a(4a+5b)}+\sqrt{b(4b+5a)}\overset{CBS}{\le}\sqrt{2(4a^2+4b^2+10ab)}=$$

$$=\sqrt{2(4(a+b)^2+2ab)}\overset{AM-GM}{\le}\sqrt{2\left(4(a+b)^2+\frac{(a+b)^2}{2}\right)}=3(a+b)$$

$$\frac{a+b}{\sqrt{a(4a+5b)}+\sqrt{b(4b+5a)}}\ge\frac{a+b}{3(a+b)}=\frac{1}{3}$$

Equality holds for $a=b$

1557. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\ge\frac{3\sqrt{2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\right)^2\ge3\sum_{cyc}\frac{bc}{\sqrt{(1+b)(1+c)}}$$

$$=3\sum_{cyc}\frac{\frac{1}{yz}}{\sqrt{\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)}}\left(x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)\overset{xyz=1}{=}3\sum_{cyc}\frac{x\sqrt{yz}}{\sqrt{(y+1)(z+1)}}$$

$$\overset{xyz=1}{=}3\sum_{cyc}\frac{x\sqrt{x}}{\sqrt{x^2(y+1)(z+1)}}\overset{Radon}{\ge}3.\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}\left(x^2(yz+y+z+1)\right)}}\overset{xyz=1}{=}$$

$$3.\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x+\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3+\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy}}$$

$$=\frac{3\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x-3+\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x-2\right)}}$$

$$\geq\frac{3\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x-3+\left(\sum_{cyc}x\right)^2+\frac{\left(\sum_{cyc}x\right)^2}{3}\left(\sum_{cyc}x-2\right)}}$$

$$\left(\because\sum_{cyc}x\overset{A-G}{\geq}3\sqrt[3]{xyz}\overset{xyz=1}{=}3\Rightarrow\sum_{cyc}x-2\geq1>0\right)\overset{?}{\geq}\frac{9}{2}$$

$$\Leftrightarrow\frac{t^3}{t-3+t^2+\frac{t^2(t-2)}{3}}\overset{?}{\geq}\frac{9}{4}\left(t=\sum_{cyc}x\right)\Leftrightarrow\frac{t^3}{3t-9+3t^2+t^2(t-2)}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow t^3-3t^2-9t+27\overset{?}{\geq}0\Leftrightarrow(t-3)^2(t+3)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\therefore\left(\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\right)^2\geq\frac{9}{2}\Rightarrow\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\geq\frac{3\sqrt{2}}{2}$$

$\forall\, a,b,c>0 \mid abc=1,''='' \text{ iff } a=b=c=1$ (QED)

1558. Let $a,b,c>0$ Prove that

$$\left(1+\frac{2}{b+c}\right)^{(b+c-2a)^3}.\left(1+\frac{2}{c+a}\right)^{(c+a-2b)^3}.\left(1+\frac{2}{a+b}\right)^{(a+b-2c)^3}.e^{-20(a^2+b^2+c^2-ab-bc-ca)}\leq1$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is equivalent to

$$(b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)+(c+a-2b)^3\ln\left(1+\frac{2}{c+a}\right)+(a+b-2c)^3\ln\left(1+\frac{2}{a+b}\right)\leq$$

$$\leq20(a^2+b^2+c^2-ab-bc-ca).$$

Using te known inequality $0\leq\ln(1+x)\leq x,\ \ \forall x\geq0$, we have

$$(b+c-2a)\ln\left(1+\frac{2}{b+c}\right)=(b+c)\ln\left(1+\frac{2}{b+c}\right)-2a\ln\left(1+\frac{2}{b+c}\right)\leq$$

$$\leq(b+c).\frac{2}{b+c}-0=2$$

$$\Rightarrow (b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)\leq 2(b+c-2a)^2 \text{ (and analogs).}$$

Therefore

$$\sum_{cyc}(b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)\leq 2\sum_{cyc}(b+c-2a)^2$$

$$=2\sum_{cyc}(4a^2+b^2+c^2-4ab+2bc-4ca)$$

$$=12(a^2+b^2+c^2-ab-bc-ca)\leq 20(a^2+b^2+c^2-ab-bc-ca),$$

as desired. Equality holds iff $a=b=c$.

1559. If $a,b,c>0$, then prove that :

$$\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2a^3}\geq\frac{a+b+c}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}-\text{RHS}=\sum_{cyc}\left(\frac{a^4}{a^3+2b^3}-\frac{a}{3}\right)=\sum_{cyc}\frac{2a(a^3-b^3)}{3(a^3+2b^3)}\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{a(a^3+2b^3-3b^3)}{a^3+2b^3}\overset{?}{\geq}0\Leftrightarrow\frac{1}{3}\sum_{cyc}a\underset{(*)}{\overset{?}{\geq}}\sum_{cyc}\frac{ab^3}{a^3+2b^3}$$

$$\text{Now,}\sum_{cyc}\frac{ab^3}{a^3+2b^3}\leq\sum_{cyc}\frac{ab^3}{ab(a+b)+b^3}=\sum_{cyc}\frac{ab^2}{a^2+ab+b^2}\overset{A-G}{\leq}\sum_{cyc}\frac{b.\frac{(a+b)^2}{4}}{\frac{3}{4}.(a+b)^2}$$

$$\left(\because a^2+ab+b^2=\frac{3}{4}.(a+b)^2+\frac{1}{4}.(a-b)^2\right)=\frac{1}{3}\sum_{cyc}a\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2a^3}\geq\frac{a+b+c}{3}\ \forall\, a,b,c>0,$$
$$''='' \text{ iff } a=b=c\text{ (QED)}$$

1560. If $a,b,c>0$, then prove that :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^2}{b}-a+b+\frac{b^2}{c}-b+c+\frac{c^2}{a}-c+a$$
$$=\frac{a^2-ab+b^2}{b}+\frac{b^2-bc+c^2}{b}+\frac{c^2-ca+a^2}{a}\rightarrow(1)$$
$$\text{Now, }\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$
$$=\sum_{cyc}\sqrt{\frac{a^2-ab+b^2}{b}.b}\overset{A-G}{\leq}\frac{1}{2}\sum_{cyc}\left(\frac{a^2-ab+b^2}{b}+b\right)$$
$$\overset{A-G}{\leq}\sum_{cyc}\frac{a^2-ab+b^2}{b}-\frac{1}{2}\sum_{cyc}\frac{2ab-ab}{b}+\frac{1}{2}\sum_{cyc}a$$
$$=\sum_{cyc}\frac{a^2-ab+b^2}{b}-\frac{1}{2}\sum_{cyc}a+\frac{1}{2}\sum_{cyc}a\overset{via\,(1)}{=}\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$
$$\therefore\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$
$$\forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1561.

If $a,b,c>0$ such that $e^{1-a-b-c}\geq(a+b+c)^{a+b+c-2}$, then

$$\left(\left(\frac{c^{1-c}}{ab}\right)^3+\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3\geq 129140163$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } f(x)=(x-2).\ln x+x-1,\quad \text{for } x>0.$$

$$\text{We have } f'(x)=\ln x+\frac{2(x-1)}{x},$$

$\forall x>0$, then f is decreasing on $(0,1]$ and increasing on $[1,\infty)$,

hence, $f(x)\geq f(1)=0$, $\forall x>0$, or $e^{1-x}\leq x^{x-2}$, $\forall x>0$, with equality for $x=1$.

But we have $e^{1-(a+b+c)}\geq(a+b+c)^{(a+b+c)-2}$, then

$e^{1-(a+b+c)}=(a+b+c)^{(a+b+c)-2}$. Therefore, $a+b+c=1$.

Now, by Hölder's inequality, we have

$$\left(\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3+\left(\frac{c^{1-c}}{ab}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3.(b+c+a)^6.(c+a+b)^6\ge$$

$$\ge\left(\sqrt[17]{\left(\frac{a^{1-a}}{bc}\right)^6.(a^{2a-2})^3.b^6.c^6}+\sqrt[17]{\left(\frac{b^{1-b}}{ca}\right)^6(b^{2b-2})^3c^6a^6}+\sqrt[17]{\left(\frac{c^{1-c}}{ab}\right)^6(c^{2c-2})^3a^6b^6}\right)^{17}$$
$$=3^{17}$$

Therefore

$$\left(\left(\frac{c^{1-c}}{ab}\right)^3+\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3\ge 3^{17}=129140163.$$

Equality holds iff $a=b=c=\frac{1}{3}$.

1562. If $a,b,c>0$, then prove that :

$$(a^2+2)(b^2+2)(c^2+2)\ge 3(a+b+c)^2+(abc-1)^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+2)(b^2+2)(c^2+2)-3(a+b+c)^2-(abc-1)^2$$
$$=a^2b^2c^2+8+4\sum_{cyc}a^2+2\sum_{cyc}a^2b^2-3\sum_{cyc}a^2-6\sum_{cyc}ab-a^2b^2c^2-1+2abc$$
$$=(abc+abc+1)+2\left((a^2b^2+1)+(b^2c^2+1)+(c^2a^2+1)\right)+\sum_{cyc}a^2-6\sum_{cyc}ab$$
$$\overset{A-G}{\ge}3\sqrt[3]{a^2b^2c^2}+2(2ab+2bc+2ca)+\sum_{cyc}a^2-6\sum_{cyc}ab\overset{?}{\ge}0$$
$$\Leftrightarrow\sum_{cyc}a^2+3\sqrt[3]{a^2b^2c^2}\overset{?}{\ge}2\sum_{cyc}ab$$
$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3+27a^2b^2c^2+9\left(\sum_{cyc}a^2\right)^2.\sqrt[3]{a^2b^2c^2}+27\left(\sum_{cyc}a^2\right).\sqrt[3]{a^4b^4c^4}$$
$$\boxed{\overset{?}{\underset{(*)}{\ge}}}8\left(\sum_{cyc}ab\right)^3$$

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow$$

$$\sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via }(1),(3)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

Via (1), (2) and (4), LHS of $(*) \geq$

$$(s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2 . \sqrt[3]{r^4s^2}$$

$$+27(s^2 - 8Rr - 2r^2). \sqrt[3]{r^8s^4} \overset{\text{Mitrinovic}}{\geq}$$

$$(s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2 . \sqrt[3]{r^4 . 27r^2}$$

$$+27(s^2 - 8Rr - 2r^2). \sqrt[3]{r^8 . 729r^4}$$

$$= (s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 27r^2(s^2 - 8Rr - 2r^2)^2 + 243r^4(s^2 - 8Rr - 2r^2)$$

$$\overset{?}{\geq} 8\left(\sum_{cyc} ab\right)^3 \overset{\text{via }(3)}{=} 8(4Rr + r^2)^3$$

$$\Leftrightarrow s^6 - (24Rr - 21r^2)s^4 + r^2(192R^2 - 336Rr + 174r^2)s^2$$

$$-r^3(1024R^3 - 960R^2r + 1272Rr^2 + 394r^3) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (8R + 2r)s^4 - r(192R^2 - 48Rr - 33r^2)s^2$$

$$+r^2(1024R^3 - 960R^2r - 24Rr^2 - 173r^3) \boxed{\overset{(***)}{\geq}} 0 \text{ and } \because$$

$$(8R + 2r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (8R + 2r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (64R^2 + 32Rr + 13r^2)s^2 \boxed{\overset{(****)}{\geq}} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3)$$

$$\text{Finally}, (64R^2 + 32Rr + 13r^2)s^2 \overset{\text{Gerretsen}}{\geq} (64R^2 + 32Rr + 13r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3) \Leftrightarrow 144r^3(R - 2r) \overset{?}{\geq} 0 \to \text{true via Euler}$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore (a^2 + 2)(b^2 + 2)(c^2 + 2)$$

$$\geq 3(a + b + c)^2 + (abc - 1)^2 \ \forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1563. If $a, b > 0$, then prove that :

$$32a^2b^2(a^3 + b^3) \leq (a^2 + b^2)(a + b)^5$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+b^2+2ab)^2 \overset{A-G}{\geq} 4(a^2+b^2)(2ab) \Rightarrow (a+b)^4 \geq 8ab(a^2+b^2)$$

$$\Rightarrow (a^2+b^2)(a+b)^5 \geq 8ab(a^2+b^2)(a+b)(a^2+b^2) \overset{?}{\geq} 32a^2b^2(a^3+b^3)$$

$$= 32a^2b^2(a+b)(a^2+b^2-ab) \Leftrightarrow (a^2+b^2)^2 \overset{?}{\geq} 4ab(a^2+b^2-ab)$$

$$\Leftrightarrow (a^2+b^2)^2 - 4ab(a^2+b^2) + 4a^2b^2 \overset{?}{\geq} 0 \Leftrightarrow (a^2+b^2-2ab)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (a-b)^4 \overset{?}{\geq} 0 \to \text{true} \therefore 32a^2b^2(a^3+b^3) \leq (a^2+b^2)(a+b)^5$$

$$\forall\, a,b > 0, '' ='' \text{ iff } a = b \text{ (QED)}$$

1564.

If $a,b,c>0$ and $\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1$, then:

$$a+b+c \geq ab+bc+ca$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1 \Rightarrow \sum_{cyc}\frac{1+a+b-(a+b)}{a+b+1} \geq 1$$

$$\Rightarrow 3 \geq 1+\sum_{cyc}\frac{a+b}{a+b+1} \Rightarrow 2 \geq \sum_{cyc}\frac{(a+b)^2}{(a+b)^2+(a+b)}$$

$$\overset{Bergstrom}{\geq} \frac{\left(2\sum_{cyc}a\right)^2}{2\sum_{cyc}a^2+2\sum_{cyc}ab+2\sum_{cyc}a}$$

$$\Rightarrow \sum_{cyc}a^2+\sum_{cyc}ab+\sum_{cyc}a \geq \left(\sum_{cyc}a\right)^2 = \sum_{cyc}a^2+2\sum_{cyc}ab \Rightarrow \sum_{cyc}a \geq \sum_{cyc}ab$$

$$\therefore a+b+c \geq ab+bc+ca$$

$$\forall\, a,b,c>0 \,\Big|\, \frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1 \text{ (QED)}$$

1565.

If $a,b,c>0$ and $2,25.\left(1+\ln^2\left(\frac{4(ab+bc+ca)}{3}\right)\right)\ge(a+b+c)^2$, then :

$$\frac{\left(2(a+b)^{a+b}-1\right)bc}{b+c}+\frac{\left(2(b+c)^{b+c}-1\right)ca}{c+a}+\frac{\left(2(c+a)^{c+a}-1\right)ab}{a+b}\ge\frac{8abc(a+b+c)^2}{3}$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\frac{4(ab+bc+ca)}{3}$. Since $(a+b+c)^2\ge 3(ab+bc+ca)=2,25.x$, then

$1+\ln^2(x)\ge x$. This inequality is true for $x\le 1$. If $x>1$,

let $f(t)=\ln(t)-\sqrt{t-1}, t\ge 1$. We have

$$f'(t)=\frac{1}{t}-\frac{1}{2\sqrt{t-1}}=-\frac{\left(\sqrt{t-1}-1\right)^2}{2t\sqrt{t-1}}\le 0,\ \ \forall t>1,\text{ then } f \text{ is strictly decreasing, hence}$$

$f(x)<f(1)=0$ or $1+\ln^2(x)<x$, which is not true.

Therefore, $x\le 1$ or $ab+bc+ca\le\frac{3}{4}$ (1)

Now, let us prove that for all $t>0$, $2t^t\ge t^2+1$ (2).

$$\text{Let } g(t)=t.\ln t-\ln\left(\frac{t^2+1}{2}\right),\ \ t>0.$$

We have $g'(t)=lnt+1-\frac{2t}{t^2+1}$ and $g''(t)=\frac{t^4+2t^3+t^2+(t-1)^2}{t(t^2+1)^2}\ge 0$, then g' is increasing and since $g'(1)=0$, then g is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence

$$g(t)\ge g(1)=0,\forall t>0\ \Leftrightarrow\ t.\ln t\ge\ln\left(\frac{t^2+1}{2}\right),\text{then } 2t^t\ge t^2+1,\forall t>0,$$

equality for $x=1$. Now, we have

$$\sum_{cyc}\frac{\left(2(a+b)^{a+b}-1\right)bc}{b+c}\overset{(2)}{\ge}\sum_{cyc}\frac{(a+b)^2bc}{b+c}=abc\sum_{cyc}\frac{(a+b)^2}{ab+ca}\overset{CBS}{\ge}\frac{4abc(a+b+c)^2}{2(ab+bc+ca)}$$

$$\overset{(1)}{\geq}\frac{8abc(a+b+c)^2}{3},$$

as desired. Equality holds iff $a=b=c=\frac{1}{2}$.

1566. If $a,b,c>0$ and $a^2+b^2+c^2=3$, then prove that :

$$\left(5a+\frac{2}{b+c}\right)^3+\left(5b+\frac{2}{c+a}\right)^3+\left(5c+\frac{2}{a+b}\right)^3\geq 648$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(5a+\frac{2}{b+c}\right)^3=\sum_{cyc}\left(a+a+a+a+a+\frac{2}{b+c}\right)^3\overset{A-G}{\geq}$$

$$\sum_{cyc}\left(6.\sqrt[6]{\frac{2a^5}{b+c}}\right)^3=216.\sqrt{2}.\sum_{cyc}\sqrt{\frac{a^8}{a^3b+a^3c}}=216.\sqrt{2}.\sum_{cyc}\frac{a^4}{\sqrt{a^3b+a^3c}}\overset{Bergstrom}{\geq}$$

$$216.\sqrt{2}.\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}\sqrt{a^3b+a^3c}}\overset{CBS}{\geq}216.\sqrt{2}.\frac{\left(\sum_{cyc}a^2\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}(a^3b+ab^3)}}\overset{a^2+b^2+c^2=3}{=}$$

$$648.\sqrt{2}.\frac{\left(\sum_{cyc}a^2\right)^2}{\sqrt{3}.\left(\sum_{cyc}a^2\right).\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-abc\sum_{cyc}a}}\overset{?}{\geq}648$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^2\right)^2\overset{?}{\underset{(*)}{\geq}}3\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-3abc\sum_{cyc}a$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{via\ (1),(3)}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$$

Via (1), (2), (3) and (4), $(*)\Leftrightarrow$

$$2(s^2-8Rr-2r^2)^2-3(s^2-8Rr-2r^2)(4Rr+r^2)+3r^2s^2\overset{(**)}{\boxed{\geq}}0 \text{ and}$$

$\because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore$ **in order to prove $(**)$, it suffices to prove :**

$$\text{LHS of }(**)\geq(s^2-16Rr+5r^2)^2\Leftrightarrow(5R-7r)s^2\overset{(***)}{\boxed{\geq}}r(72R^2-108Rr+9r^2)$$

$$\text{Now, }(5R-7r)s^2\overset{\text{Gerretsen}}{\geq}(5R-7r)(16Rr-5r^2)\overset{?}{\geq}r(72R^2-108Rr+9r^2)$$

$$\Leftrightarrow 8R^2-29Rr+26r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(8R-13r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\left(5a+\frac{2}{b+c}\right)^3+\left(5b+\frac{2}{c+a}\right)^3+\left(5c+\frac{2}{a+b}\right)^3$$

$$\geq 648\ \forall a,b,c>0\ \middle|\ a^2+b^2+c^2=3,\ ''=''\text{ iff }a=b=c=1\ \text{(QED)}$$

Solution 2 by Tapas Das-India

$$\textit{Note: Vasc's inequality }(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^3a)$$

$$\left(5a+\frac{2}{b+c}\right)^3+\left(5b+\frac{2}{c+a}\right)^3+\left(5c+\frac{2}{a+b}\right)^3\overset{AM-GM}{\geq}$$

$$\geq\left(\sum\left(6\sqrt[6]{\frac{2a^5}{b+c}}\right)\right)^3=216.\sqrt{2}\left(\sum\frac{a^{\frac{5}{2}}}{(b+c)^{\frac{1}{2}}}\right)=$$

$$=216.\sqrt{2}\left(\sum\frac{a^{\frac{5}{2}}a^{\frac{3}{2}}}{a^{\frac{3}{2}}(b+c)^{\frac{1}{2}}}\right)=216.\sqrt{2}\left(\sum\frac{a^4}{\sqrt{a^3(b+c)}}\right)\overset{\textit{Bergstrom \& CBS}}{\geq}$$

$$\geq 216.\sqrt{2}\frac{(\sum a^2)^2}{\left(3\left((a^3b+b^3c+c^3a)+(a^3c+c^3b+b^3a)\right)\right)^{\frac{1}{2}}}\overset{Vasc}{\geq}$$

$$216.\sqrt{2}\frac{(\sum a^2)^2}{\sqrt{2}\,(\sum a^2)}=216.\left(\sum a^2\right)=216.3=648$$

1567. If $a,b,c\geq 0$, then prove that :

$$(a^{2024}-a^{2022}+3)(b^{2024}-b^{2022}+3)(c^{2024}-c^{2022}+3)\geq 9(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^{2024}-a^{2022}+3\overset{?}{\geq}a^2+2\Leftrightarrow a^{2022}(a^2-1)\overset{?}{\geq}a^2-1$$
$$\Leftrightarrow(a^2-1)\left((a^2)^{1011}-1\right)\overset{?}{\geq}0\Leftrightarrow(a^2-1)^2\left((a^2)^{1010}+(a^2)^{1009}+\cdots+1\right)\overset{?}{\geq}0$$
$$\to\text{true}\therefore a^{2024}-a^{2022}+3\geq a^2+2\text{ and analogs}\Rightarrow$$
$$\text{LHS}\geq(a^2+2)(b^2+2)(c^2+2)\therefore\text{it suffices to prove :}$$
$$(a^2+2)(b^2+2)(c^2+2)\boxed{\overset{(\bullet)}{\geq}}9(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ $a=b=c=0$ and then : LHS of $(\bullet)=8$ and RHS of $(\bullet)=0\Rightarrow$ LHS $>$ RHS

$\boxed{\text{Case 2}}$ Exactly two among a,b,c equal zero and WLOG we may assume $b=c=0$ and then : LHS of $(\bullet)=4(a^2+2)$ and RHS of $(\bullet)=0\Rightarrow$ LHS $>$ RHS

$\boxed{\text{Case 3}}$ Exactly one among a,b,c equals zero and WLOG we may assume $a=0$ $(b,c>0)$ and then : LHS of $(\bullet)-$ RHS of $(\bullet)=2(b^2+2)(c^2+2)-9bc$
$$=2b^2c^2+4(b^2+c^2)+8-9bc\overset{A-G}{\geq}2b^2c^2+8bc+8-9bc$$
$$=(bc-1)^2+b^2c^2+bc+7>0\Rightarrow\text{LHS}>\text{RHS}$$

$\boxed{\text{Case 4}}$ $a,b,c>0$ and LHS of $(\bullet)=a^2b^2c^2+2+2\sum_{cyc}a^2b^2+4\sum_{cyc}a^2+8$
$$=(a^2b^2c^2+1+1)+2\left((a^2b^2+1)+(b^2c^2+1)+(c^2a^2+1)\right)+4\sum_{cyc}a^2$$
$$\overset{A-G}{\geq}3\sqrt[3]{a^2b^2c^2}+4\sum_{cyc}ab+4\sum_{cyc}a^2\overset{?}{\geq}9(ab+bc+ca)$$
$$\Leftrightarrow 4\sum_{cyc}a^2+3\sqrt[3]{a^2b^2c^2}\boxed{\underset{(\bullet\bullet)}{\overset{?}{\geq}}}5\sum_{cyc}ab$$
$$\text{Now,}\sum_{cyc}a^2+3\sqrt[3]{a^2b^2c^2}\overset{?}{\geq}2\sum_{cyc}ab$$
$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3+27a^2b^2c^2+9\left(\sum_{cyc}a^2\right)^2.\sqrt[3]{a^2b^2c^2}+27\left(\sum_{cyc}a^2\right).\sqrt[3]{a^4b^4c^4}$$
$$\boxed{\underset{(*)}{\overset{?}{\geq}}}8\left(\sum_{cyc}ab\right)^3$$

Assigning $b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);
$$\text{so }2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x,b=s-y,c=s-z$$
$$\therefore abc=r^2s\to(2)\text{and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via }(1),(3)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

Via (1), (2) and (4), LHS of $(*) \ge (s^2 - 8Rr - 2r^2)^3$

$$+27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2.\sqrt[3]{r^4s^2} + 27(s^2 - 8Rr - 2r^2).\sqrt[3]{r^8s^4} \overset{\text{Mitrinovic}}{\ge}$$

$$(s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2.\sqrt[3]{r^4.27r^2}$$

$$+27(s^2 - 8Rr - 2r^2).\sqrt[3]{r^8.729r^4}$$

$$= (s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 27r^2(s^2 - 8Rr - 2r^2)^2 + 243r^4(s^2 - 8Rr - 2r^2)$$

$$\overset{?}{\ge} 8\left(\sum_{cyc} ab\right)^3 \overset{\text{via }(3)}{=} 8(4Rr + r^2)^3$$

$$\Leftrightarrow s^6 - (24Rr - 21r^2)s^4 + r^2(192R^2 - 336Rr + 174r^2)s^2$$

$$-r^3(1024R^3 - 960R^2r + 1272Rr^2 + 394r^3) \boxed{\underset{(**)}{\overset{?}{\ge}}} 0 \text{ and}$$

$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove $(**)$, it suffices to prove :

$$: \text{LHS of } (**) \ge (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow$$

$$(8R + 2r)s^4 - r(192R^2 - 48Rr - 33r^2)s^2$$

$$+r^2(1024R^3 - 960R^2r - 24Rr^2 - 173r^3) \boxed{\overset{(***)}{\ge}} 0$$

and $\because (8R + 2r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove $(***)$,

it suffices to prove : LHS of $(***) \ge (8R + 2r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (64R^2 + 32Rr + 13r^2)s^2 \boxed{\overset{(****)}{\ge}} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3)$$

Finally, $(64R^2 + 32Rr + 13r^2)s^2 \overset{\text{Gerretsen}}{\ge} (64R^2 + 32Rr + 13r^2)(16Rr - 5r^2)$

$\overset{?}{\ge} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3) \Leftrightarrow 144r^3(R - 2r) \overset{?}{\ge} 0 \to$ true via Euler

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \ge 2\sum_{cyc} ab$$

$$\Rightarrow 4\sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \ge 2\sum_{cyc} ab + 3\sum_{cyc} a^2 \ge 5\sum_{cyc} ab \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$\therefore$ combining *all* cases, $(\bullet)$ is true $\forall a, b, c \ge 0$

$\therefore (a^{2024} - a^{2022} + 3)(b^{2024} - b^{2022} + 3)(c^{2024} - c^{2022} + 3) \ge 9(ab + bc + ca)$

$\forall a, b, c \ge 0$, " = " iff $a = b = c = 1$ (QED)

1568. If $x, y > 0$ and $xy = 1$, then prove that :

$$x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1} \ge 11$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1}$$

$$= (x + y)^2 - 2xy + 3(x + y) + \frac{9}{(x + y)^2 - 2xy + 1} \overset{xy=1}{=} t^2 - 2 + 3t + \frac{9}{t^2 - 2 + 1}$$

$$(t = x + y) = \frac{(t^2 - 1)(t^2 - 2 + 3t) + 9}{t^2 - 1} \overset{?}{\ge} 11$$

$$\Leftrightarrow (t^2 - 1)(t^2 - 2 + 3t) + 9 \overset{?}{\ge} 11(t^2 - 1) \Leftrightarrow t^4 + 3t^3 - 14t^2 - 3t + 22 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t - 2)\left((t - 2)(t^2 + 7t + 10) + 9\right) \overset{?}{\ge} 0 \to \text{true} \because t = x + y \overset{A-G}{\ge} 2\sqrt{xy} = 2$$

$$\therefore x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1} \ge 11$$

$$\forall\, x, y > 0 \,\big|\, xy = 1, '' ='' \text{ iff } x = y = 1 \text{ (QED)}$$

1569. If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{2x}{x^6 + y^4} + \frac{2y}{y^6 + z^4} + \frac{2z}{z^6 + x^4} \le \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{2x}{x^6 + y^4} + \frac{2y}{y^6 + z^4} + \frac{2z}{z^6 + x^4} \overset{AM-GM}{\le} \sum \frac{2x}{2x^3y^2} =$$

$$= \sum \frac{1}{x^2y^2} = \frac{x^2 + y^2 + z^2}{x^2y^2z^2} = \frac{x^2y^2z^2(x^2 + y^2 + z^2)}{x^4y^4z^4} =$$

$$= \frac{\sum (x^2y^2)(x^2z^2)}{x^4y^4z^4} \overset{(\sum ab) \le (\sum a^2) \forall\, a,b,c>0}{\le}$$

$$\le \frac{x^4y^4+y^4z^4+z^4x^4}{x^4y^4z^4}=\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}$$

Equality holds for $x=y=z=1$.

1570. If $x,y,z>0$, then prove that :

$$27(x+y)^4(y+z)^4(z+x)^4\ge 4096x^3y^3z^3(x+y+z)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(x+y)(y+z)(z+x)\overset{?}{\ge}\frac{8}{9}(x+y+z)(xy+yz+zx)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-xyz\right)\overset{?}{\ge}8\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\Leftrightarrow$$

$$\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\ge}9xyz\to\text{true}\because\sum_{cyc}x\overset{A-G}{\ge}3\sqrt[3]{xyz}\text{ and }\sum_{cyc}xy\overset{A-G}{\ge}3\sqrt[3]{x^2y^2z^2}\Rightarrow$$

$$\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\ge 9xyz\therefore(x+y)(y+z)(z+x)\ge\frac{8}{9}(x+y+z)(xy+yz+zx)$$

$$\Rightarrow 27(x+y)^4(y+z)^4(z+x)^4\ge\frac{27.4096}{9^4}\left(\sum_{cyc}x\right)^4\left(\sum_{cyc}xy\right)^4$$

$$\ge\frac{27.9xyz.4096}{9^4}\left(\sum_{cyc}x\right)^3\left(\sum_{cyc}xy\right)^3\overset{A-G}{\ge}\frac{27.9xyz.4096}{9^4}\left(\sum_{cyc}x\right)^3(27x^2y^2z^2)$$

$$\Rightarrow 27(x+y)^4(y+z)^4(z+x)^4\ge 4096x^3y^3z^3(x+y+z)^3\ \forall\, x,y,z>0,$$

$$''=''\text{ iff } x=y=z\ (\text{QED})$$

1571. If $a,b,c\in\mathbb{R}$ and $a^2+b^2+c^2\le 2$, then prove that :

$$2024ca-ab-bc\ge -2024$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2024ca + 2024 - ab - bc \overset{2 \ge a^2+b^2+c^2}{\ge}$$

$$2024ca + 1012(a^2 + b^2 + c^2) - b(c + a) =$$

$$1012(c^2 + a^2 + 2ca) - b(c + a) + 1012b^2 = 1012(c + a)^2 - b(c + a) + 1012b^2$$

$$\ge 0 \begin{pmatrix} \because 1012(c + a)^2 - b(c + a) + 1012b^2 \text{ is a quadratic polynomial in } "b" \text{ or } "c + a" \\ \text{with discriminant} = (1 - 4.1012^2)(c + a)^2 \text{ or } (1 - 4.1012^2)b^2 \le 0 \end{pmatrix}$$

$$\text{with equality for } b = c + a = 0 \therefore 2024ca - ab - bc \ge -2024 \; \forall \, a, b, c \in \mathbb{R}$$

$$\Big| \, a^2 + b^2 + c^2 \le 2, "=" \text{ iff } (a = 1, b = 0, c = -1) \text{ or } (a = -1, b = 0, c = 1) \text{ (QED)}$$

1572. If $x, y, z \in [-2, 2]$, then prove that :

$$2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$-2 \le x \le 2 \Rightarrow (x + 2)(x - 2) \le 0 \Rightarrow x^2 - 4 \le 0 \Rightarrow 4 - x^2 \ge 0$ and similarly, $4 - y^2 \ge 0$ and $4 - z^2 \ge 0$ and setting $a = 4 - x^2, b = 4 - y^2$, $c = 4 - z^2$, we notice that : $\boxed{0 \le a, b, c \le 4}$ $(\because a = 4 - x^2 \le 4$ and analogs$)$ and $x^2 = 4 - a, y^2 = 4 - b, z^2 = 4 - c$ and via such substitutions,

$$2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192$$

$$\Leftrightarrow 2\sum_{cyc}(4 - a)^3 - \sum_{cyc}\Big((4 - a)^2(4 - b)\Big) - 192 \le 0$$

$$\Leftrightarrow \sum_{cyc} a^2b + 20\sum_{cyc} a^2 \overset{(*)}{\le} 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a$$

Now, $(a - 4)^2 \ge 0 \Rightarrow a^2 + 16 \ge 8a \Rightarrow 2a^3 + 32a \ge 16a^2$ $(\because a \ge 0)$ and analogs

$$\Rightarrow 2\sum_{cyc} a^3 + 32\sum_{cyc} a \ge 16\sum_{cyc} a^2 \to (1)$$

Also, $16\sum_{cyc} a \ge 4\sum_{cyc} a^2 \to (2)$ $\big(\because 4 \ge a \Rightarrow 4a \ge a^2$ $(\because a \ge 0)$ and analogs$\big)$ and

moreover, $4\sum_{cyc} ab \ge a^2b + b^2c + c^2a$ $(\because 4 \ge a, b, c$ and $ab, bc, ca \ge 0)$

$$\Rightarrow 8\sum_{cyc} ab \ge \sum_{cyc} a^2b + 4\sum_{cyc} ab \overset{ab,bc,ca \ge 0 \Rightarrow \sum_{cyc} ab \ge 0}{\ge} \sum_{cyc} a^2b \to (3)$$

$$\therefore (1)+(2)+(3) \Rightarrow 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a \ge \sum_{cyc} a^2 b + 20\sum_{cyc} a^2$$

$$\Rightarrow (*) \text{ is true } \therefore 2(x^6+y^6+z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192\ \forall\, x,y,z \in [-2,2],$$

$$'' ='' \ (x=2,y=2,z=2) \text{ or } (x=-2,y=-2,z=-2) \text{ or } (x=2,y=2,z=-2)$$

$$\text{or } (x=2,y=-2,z=2) \text{ or } (x=2,y=-2,z=-2) \text{ or } (x=-2,y=2,z=2)$$

$$\text{or } (x=-2,y=2,z=-2) \text{ or } (x=-2,y=-2,z=2)\ (\text{QED})$$

1573. Find all values of k such that

$$\sum_{cyc} \frac{a^2}{b^2+c^2} + \sum_{cyc} \sqrt{\frac{a}{b+c}} \ge k \sum_{cyc} \frac{a}{b+c}$$

is true for all $a,b,c > 0$.

Proposed by Nguyen Van Canh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

For $b=c=1$, we have $k \le \dfrac{\frac{a^2}{2}+\frac{2}{a^2+1}+\sqrt{\frac{a}{2}}+\frac{2}{\sqrt{a+1}}}{\frac{a}{2}+\frac{2}{a+1}} \underbrace{\to}_{a\to 0^+} 2.$

Let us prove that the given inequality is true for $k=2$.

WLOG, we assume that $ab+bc+ca=1$. We have

$$\sum_{cyc} \sqrt{\frac{a}{b+c}} = \sum_{cyc} \frac{2a}{2\sqrt{a(b+c)}} \overset{AM-GM}{\ge} \sum_{cyc} \frac{2a}{a+(b+c)} = 2 = 2(ab+bc+ca)^2 \quad (1)$$

Now, by AM − GM inequality, we have

$$\frac{a^2}{b^2+c^2} + a^2(b^2+c^2) \ge 2a^2 \Rightarrow \frac{a^2}{b^2+c^2} \ge 2a^2 - a^2(b^2+c^2) \text{ (and analogs)}$$

$$\Rightarrow \sum_{cyc} \frac{a^2}{b^2+c^2} \ge 2(a^2+b^2+c^2) - 2(a^2b^2+b^2c^2+c^2a^2) \quad (2)$$

From the results (1) and (2), it suffices to prove that

$$a^2+b^2+c^2+2abc(a+b+c) \ge \sum_{cyc} \frac{a}{b+c}.$$

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a(ab+bc+ca)}{b+c}=\sum_{cyc}a^2+abc\sum_{cyc}\frac{1}{b+c}=\sum_{cyc}a^2+abc\sum_{cyc}\left(a+\frac{bc}{b+c}\right)$$

$$\overset{HM-AM}{\underbrace{\leq}}\sum_{cyc}a^2+abc\sum_{cyc}\left(a+\frac{b+c}{4}\right)\leq a^2+b^2+c^2+2abc(a+b+c).$$

So the given inequality is true for

$k=2$. Therefore, the given inequality is true for all $\boxed{k\leq 2}$.

1574. Find all values of k such that

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}+\sum_{cyc}\frac{a^3}{b^3+c^3}\geq k\sum_{cyc}\frac{a^2}{b^2+c^2},$$

is true for all $a,b,c>0$.

Proposed by Nguyen Van Canh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

For $b=c=1$, we have

$$k\leq\frac{\sqrt{\frac{a}{2}}+\frac{2}{\sqrt{a+1}}+\frac{a^3}{2}+\frac{2}{a^3+1}}{\frac{a^2}{2}+\frac{2}{a^2+1}}\underset{a\to 0^+}{\underbrace{\to}}2.$$ We will prove that the values of k are

$k\leq 2$. For this, it is suffices to prove that the given inequality is true for $k=2$.

We have: $$\sum_{cyc}\sqrt{\frac{a}{b+c}}=2\sum_{cyc}\frac{a}{2\sqrt{a(b+c)}}\overset{AM-GM}{\underbrace{\geq}}2\sum_{cyc}\frac{a}{a+(b+c)}=2.$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}=1+\frac{a^6+b^6+c^6+a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}=1+\frac{a^6+b^6+c^6+a^2b^2c^2}{\sum_{cyc}a^2b^2(a^2+b^2)+2a^2b^2c^2}\leq$$

$$\leq 1+\frac{a^6+b^6+c^6+a^2b^2c^2}{2(a^3b^3+b^3c^3+c^3a^3)+2a^2b^2c^2}.$$

From these results, it suffices to prove that

$$(a^3b^3+b^3c^3+c^3a^3+a^2b^2c^2)\sum_{cyc}\frac{a^3}{b^3+c^3}\geq a^6+b^6+c^6+a^2b^2c^2$$

$$\Leftrightarrow a^3b^3c^3\sum_{cyc}\frac{1}{b^3+c^3}+a^2b^2c^2\sum_{cyc}\frac{a^3}{b^3+c^3}\ge a^2b^2c^2,$$

which is true because $\sum_{cyc}\frac{a^3}{b^3+c^3}\ge\sum_{cyc}\frac{a^3}{a^3+b^3+c^3}=1.$

Therefore, the values of k such that the given inequality is true for all $a,b,c>0$ are

$$k\le 2.$$

1575. If $x_k>0$ $(k=1,2,\dots,n)$, then prove that :

$$\sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4+x_1^2x_2^2+x_2^4\right)}\ge\sum_{k=1}^{n}x_k^2\ge\sum_{cyc}x_1\sqrt{\frac{1}{3}\left(2x_1^2+x_2x_3\right)}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4+x_1^2x_2^2+x_2^4\right)}\ge\sum_{cyc}\sqrt{\frac{1}{3}\cdot\frac{3}{4}\cdot\left(x_1^2+x_2^2\right)^2}=\frac{1}{2}\sum_{cyc}\left(x_1^2+x_2^2\right)$$

$$=\sum_{k=1}^{n}x_k^2\ \text{and again,}\ \sum_{cyc}x_1\sqrt{\frac{1}{3}\left(2x_1^2+x_2x_3\right)}\overset{A-G}{\le}\sum_{cyc}\frac{x_1^2+\frac{2x_1^2+x_2x_3}{3}}{2}$$

$$=\frac{5}{6}\sum_{k=1}^{n}x_k^2+\frac{1}{6}\sum_{cyc}x_2x_3\overset{A-G}{\le}\frac{5}{6}\sum_{k=1}^{n}x_k^2+\frac{1}{12}\sum_{cyc}\left(x_2^2+x_3^2\right)$$

$$=\frac{5}{6}\sum_{k=1}^{n}x_k^2+\frac{1}{6}\sum_{k=1}^{n}x_k^2=\sum_{k=1}^{n}x_k^2$$

$$\therefore\sum_{cyc}\sqrt{\frac{1}{3}\left(x_1^4+x_1^2x_2^2+x_2^4\right)}\ge\sum_{k=1}^{n}x_k^2\ge\sum_{cyc}x_1\sqrt{\frac{1}{3}\left(2x_1^2+x_2x_3\right)}$$

$$\forall x_k>0\ (k=1,2,\dots,n),''=''\ \text{iff}\ x_1=x_2=\dots=x_n\ \text{(QED)}$$

1576. If $a_k > 0$ $(k = 1, 2, \dots, n)$, then prove that :

$$\sum_{cyc}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}} \dots + \frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right) \geq 3$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

We assign $a_1 = a, a_2 = b, a_3 = c$ and then :

$$\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}$$

$$=\sum_{\substack{cyc\\a,b,c}}\frac{a}{\sqrt{b^2+(n^2-2)bc+c^2}}=\sum_{\substack{cyc\\a,b,c}}\frac{a^2}{\sqrt{a}.\sqrt{ab^2+(n^2-2)abc+ac^2}}\overset{\text{Bergstrom}}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}a^2b+\sum_{cyc}ab^2+3(n^2-2)abc}}=$$

$$=\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+3(n^2-3)abc}}\geq$$

$$\geq\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\left(\sum_{cyc}a\right).\frac{\left(\sum_{cyc}a\right)^2}{3}+\frac{3(n^2-3)}{27}\left(\sum_{cyc}a\right)^3}}=\frac{1}{\sqrt{\frac{1}{3}+\frac{n^2-3}{9}}}=\frac{3}{n}\therefore$$

$$\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}$$

$$\geq\frac{3}{n}\text{ and analogs}\therefore\sum_{cyc}\left(\frac{a_1}{\sqrt{a_2^2+(n^2-2)a_2a_3+a_3^2}}+\frac{a_2}{\sqrt{a_3^2+(n^2-2)a_3a_1+a_1^2}}+\frac{a_3}{\sqrt{a_1^2+(n^2-2)a_1a_2+a_2^2}}\right)$$

$$\geq n.\frac{3}{n}=3,''=''\text{ iff }a_1=a_2=\dots=a_n\text{ (QED)}$$

1577. If $a,b>0$ and $a^3+b^3=ab$, then prove that :

$$\frac{1}{a^6}+\frac{1}{b^6}\geq 128$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^6}+\frac{1}{b^6}=\frac{a^6+b^6}{a^6b^6}=\frac{(a^3+b^3)^2-2a^3b^3}{a^6b^6}\overset{a^3+b^3=ab}{=}\frac{a^2b^2-2a^3b^3}{a^6b^6}$$

$$=\frac{1-2ab}{a^4b^4}\overset{?}{\geq}128\Leftrightarrow 128t^4+2t-1\overset{?}{\leq}0\ (t=ab)$$

$$\Leftrightarrow(4t-1)(32t^3+8t^2+2t+1)\overset{?}{\leq}0\to\text{true}\because ab=a^3+b^3\overset{A-G}{\geq}2\sqrt{a^3b^3}$$

$$\Rightarrow t^2 \geq 4t^3 \Rightarrow t \leq \frac{1}{4} \Rightarrow 4t-1 \leq 0 \therefore \frac{1}{a^6}+\frac{1}{b^6} \geq 128$$

$$\forall a,b > 0 \,\middle|\, a^3+b^3 = ab, ''='' \text{ iff } a = b = \frac{1}{2} \text{ (QED)}$$

1578. **If $a,b,c > 0$ and $a+b+c = 1$, then prove that :**

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq 3(a^2+b^2+c^2)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr+r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via }(1),(3)}{=}$$

$$s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2 = s^2-8Rr-2r^2 \rightarrow (4)$$

$$\text{Now, } \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} = \sum_{cyc}\left(\frac{a^2}{b}+b+2a\right)-3\sum_{cyc} a = \sum_{cyc}\frac{(a+b)^2}{b}-3\sum_{cyc} a$$

$$= \sum_{cyc}\frac{\left((a+b)(a+c)\right)^2}{b(c+a)^2}-3\sum_{cyc} a \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2+3\sum_{cyc} ab\right)^2}{\sum_{cyc} bc^2+\sum_{cyc} a^2 b+6abc}-3\sum_{cyc} a$$

$$= \frac{\left(\left(\sum_{cyc} a\right)^2+\sum_{cyc} ab\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-3abc+6abc}-3\sum_{cyc} a \overset{\text{via }(1),(2),(3)}{=} \frac{\left(s^2+4Rr+r^2\right)^2}{s(4Rr+r^2)+3r^2 s}-3s$$

$$= \frac{\left(s^2+4Rr+r^2\right)^2-3s^2(4Rr+r^2)-9r^2s^2}{s(4Rr+4r^2)} \overset{?}{\geq} 3(a^2+b^2+c^2) \overset{a+b+c=1}{=} \frac{3\sum_{cyc} a^2}{\sum_{cyc} a}$$

$$\overset{\text{via }(1),(4)}{=} \frac{3(s^2-8Rr-2r^2)}{s} \Leftrightarrow \left(s^2+4Rr+r^2\right)^2-3s^2(4Rr+r^2)-9r^2s^2$$

$$\overset{?}{\geq} 3(s^2-8Rr-2r^2)(4Rr+4r^2)$$

$$\Leftrightarrow s^4-(16Rr+22r^2)s^2+r^2(112R^2+128Rr+25r^2) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}(16Rr-5r^2)s^2-(16Rr+22r^2)s^2$$

$$+r^2(112R^2+128Rr+25r^2)\overset{?}{\geq}0\Leftrightarrow 112R^2+128Rr+25r^2\boxed{\underset{(**)}{\overset{?}{\geq}}}27s^2$$

$$\text{Again, }27s^2\overset{\text{Gerretsen}}{\leq}108R^2+108Rr+81r^2\overset{?}{\leq}112R^2+128Rr+25r^2$$

$$\Leftrightarrow R^2+5Rr-14r^2\overset{?}{\geq}0\Leftrightarrow(R^2-4r^2)+5r(R-2r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3(a^2+b^2+c^2)$$

$$\forall\, a,b,c>0\,\middle|\,a+b+c=1,''=''\text{ iff }a=b=c=\frac{1}{3}\text{ (QED)}$$

Solution 2 by Tapas Das-India

$$(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq 3(a^2+b^2+c^2)\,or$$

$$\sum\frac{a^3}{b}+\sum\frac{ab^2}{c}\geq 2\sum a^2\ or\ \sum\frac{a^4}{ab}+\sum\frac{a^2b^2}{ac}\geq 2\sum a^2\ or$$

$$\frac{(\sum a^2)^2}{\sum ab}+\frac{(\sum ab)^2}{\sum ab}-2\sum a^2\geq 0(Bergstrom)\ or$$

$$\left(\sum a^2\right)^2-2\sum ab.\sum a^2+\left(\sum ab\right)^2\geq 0\ or\left(\sum a^2-\sum ab\right)^2\geq 0\ true$$

$$,Now\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}=3(a^2+b^2+c^2)(as\ a+b+c=1)$$

1579. If $a,b,c>0$, then prove that :

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\leq\frac{3(a^2+b^2+c^2)}{a+b+c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so }2\sum_{\text{cyc}}a=\sum_{\text{cyc}}x=2s\Rightarrow\sum_{\text{cyc}}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z\text{ and}$$

$$\text{such substitutions}\Rightarrow\sum_{\text{cyc}}ab=\sum_{\text{cyc}}(s-x)(s-y)\Rightarrow\sum_{\text{cyc}}ab=4Rr+r^2\to(2),$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{via\ (1),(2)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3)$$

$$\text{Now, } \frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c}$$

$$\Leftrightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a} \ge \sum_{cyc} \frac{(b+c)^2 - 2bc}{b+c} \Leftrightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a} + 2\sum_{cyc} \frac{bc}{b+c} \ge 2\sum_{cyc} a \to (i)$$

$$\sum_{cyc} \frac{bc}{b+c} = \sum_{cyc} \frac{(s-y)(s-z)}{x} = \sum_{cyc} \frac{s^2 - s(2s-x) + yz}{x}$$

$$= \frac{-s^2}{4Rrs}\sum_{cyc} xy + 3s + \frac{1}{4Rrs}\sum_{cyc} x^2y^2$$

$$= \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2 - s^2(s^2+4Rr+r^2) + 12Rrs^2}{4Rrs}$$

$$= \frac{r(4R+r)^2 + rs^2}{4Rs} \Rightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a} + 2\sum_{cyc} \frac{bc}{b+c} - 2\sum_{cyc} a$$

$$\overset{via\ (1)\ and\ (3)}{=} \frac{3(s^2 - 8Rr - 2r^2)}{s} + \frac{r(4R+r)^2 + rs^2}{2Rs} - 2s$$

$$= \frac{6R(s^2 - 8Rr - 2r^2) + r(4R+r)^2 + rs^2 - 4Rs^2}{2Rs} \overset{?}{\ge} 0$$

$$\Leftrightarrow (2R+r)s^2 \underset{(*)}{\overset{?}{\ge}} r(32R^2 + 4Rr - r^2)$$

$$\text{Now, } (2R+r)s^2 \overset{Gerretsen}{\ge} (2R+r)(16Rr - 5r^2) \overset{?}{\ge} r(32R^2 + 4Rr - r^2)$$

$$\Leftrightarrow 2Rr - 4r^2 \overset{?}{\ge} 0 \Leftrightarrow 2r(R-2r) \overset{?}{\ge} 0 \to \text{true via Euler} \Rightarrow (*) \Rightarrow (i) \text{ is true}$$

$$\therefore \frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c}$$

$\forall a, b, c > 0,$ " = " iff $a = b = c$ (QED)

1580. If $x, y, z \in [0,2]$, then prove that :

$$2(x+y+z) - (xy+yz+zx) \le 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $x = y = z = 0$ and then : LHS $= 0 < 4$

Case 2 Exactly 2 among x, y, z equal zero and WLOG we may assume $y = z = 0$ and $0 \le x \le 2$ and then : LHS $-$ RHS $= 2x - 4 = 2(x-2) \le 0$

$$\Rightarrow 2(x+y+z)-(xy+yz+zx)\le 4$$

Case 3 Exactly one among x,y,z equals zero and WLOG we may assume $x=0$

and $0\le y,z\le 2$ and then : $(y-2)(z-2)\ge 0\Rightarrow yz-2y-2z+4\ge 0\to (1)$

Also, $LHS-RHS=4+yz-2y-2z\overset{via\,(1)}{\ge}0\Rightarrow 2(x+y+z)-(xy+yz+zx)\le 4$

Case 4 $0<x,y,z\le 2$ and $\because 0<x\le 2\therefore \frac{1}{x}\ge\frac{1}{2}\Rightarrow \frac{1}{x}-\frac{1}{2}=a\,(say)\ge 0\Rightarrow x=$ $\frac{2}{2a+1}$ and similarly, setting $\frac{1}{y}-\frac{1}{2}=b\ge 0$ and $\frac{1}{z}-\frac{1}{2}=c\ge 0$, we get : $y=\frac{2}{2b+1}$ and $z=\frac{2}{2c+1}$ and then : $2(x+y+z)-(xy+yz+zx)\le 4\Leftrightarrow$

$$4\sum_{cyc}\frac{1}{2a+1}-4\sum_{cyc}\left(\frac{1}{2b+1}\cdot\frac{1}{2c+1}\right)\le 4$$

$$\Leftrightarrow \prod_{cyc}(2a+1)+\sum_{cyc}(2a+1)-\sum_{cyc}\big((2b+1)(2c+1)\big)\ge 0$$

$\Leftrightarrow 8abc+1\ge 0\to$ true $\because a,b,c\ge 0\therefore 2(x+y+z)-(xy+yz+zx)<4$ and

$\therefore$ combining all cases, $2(x+y+z)-(xy+yz+zx)\le 4\ \forall\, x,y,z\in[0,2]$,

" = " iff $(x=2,y=z=0)$ or $(y=2,z=x=0)$ or $(z=2,x=y=0)$ or $(x=y=2,z=0)$ or $(y=z=2,x=0)$ or $(z=x=2,y=0)$ (QED)

1581. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$LHS-RHS\overset{a+b+c=3}{=}\sum_{cyc}\frac{4}{(b+c)^3}-\sum_{cyc}\frac{3-(b+c)}{b+c}=\sum_{cyc}\left(\frac{4}{x^3}-\frac{3}{x}+1\right)$$

$$(x=b+c,y=c+a,z=a+b)=\sum_{cyc}\frac{x^3-3x^2+4}{x^3}=\sum_{cyc}\frac{(x-2)^2(x+1)}{x^3}\ge 0$$

$$\therefore\frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$\forall\, a,b,c>0 \,\big|\, a+b+c=3, '' = ''$ iff $a=b=c=1$ (QED)

Solution 2 by Tapas Das-India

$$\frac{a}{b+c}=4.\frac{a}{b+c}.\frac{1}{2}.\frac{1}{2}\overset{AM-GM}{\le}\frac{4}{27}\left(\frac{a}{b+c}+\frac{1}{2}+\frac{1}{2}\right)^3=$$

$$=\frac{4}{27}\left(\frac{a}{b+c}+1\right)^3=\frac{4}{27}\left(\frac{a+b+c}{b+c}\right)^3=4\left(\frac{1}{(b+c)^3}\right)\ since\ a+b+c=3.$$

$$similarly,\frac{b}{c+a}\le 4.\frac{1}{(c+a)^3}\ and\ \frac{c}{a+b}\le 4.\frac{1}{(a+b)^3}$$

using this result we get

$$\frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

1582. If $a,b,c\ge 0$ and $a+b+c=1$, then prove that :

$$ab+bc+ca-3abc\le\frac{1}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 among a,b,c equal zero and WLOG we may assume

$$b=c=0\ (a=1)\ \text{and then : LHS}=0<\frac{1}{4}$$

$\boxed{\text{Case 2}}$ Exactly 1 among a,b,c equals zero and WLOG we may

$$assume\ a=0\ (b+c=1)\ \text{and then : LHS}=bc\overset{A-G}{\le}\frac{(b+c)^2}{4}=\frac{1}{4}$$

$$\boxed{\text{Case 3}}\ a,b,c>0\ \text{and then : LHS}\overset{a+b+c=1}{=}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)-3abc$$

$$\overset{?}{\le}\frac{1}{4}\overset{a+b+c=1}{=}\frac{1}{4}\left(\sum_{cyc}a\right)^3$$

$$\Leftrightarrow\sum_{cyc}a^2b+\sum_{cyc}ab^2\overset{?}{\le}\frac{1}{4}\left(\sum_{cyc}a^3+3\left(2abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)$$

$$\Leftrightarrow \sum_{cyc} a^3 + 6abc + 3\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \overset{?}{\geq} 4\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)$$

$$\Leftrightarrow \sum_{cyc} a^3 + 6abc \overset{?}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \to \text{true}$$

$$\because \sum_{cyc} a^3 + 6abc \overset{\text{Schur}}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 + 3abc > \sum_{cyc} a^2b + \sum_{cyc} ab^2$$

$$\Rightarrow ab + bc + ca - 3abc < \frac{1}{4} \text{ and combining } all \text{ cases}, ab + bc + ca - 3abc \leq \frac{1}{4}$$

$$\forall\, a, b, c \geq 0 \,\big|\, a + b + c = 1, '' ='' \text{ iff } \left(a = 0, b = c = \frac{1}{2}\right)$$

$$\text{or } \left(b = 0, c = a = \frac{1}{2}\right) \text{ or } \left(c = 0, a = b = \frac{1}{2}\right) \text{ (QED)}$$

1583. If $a, b, c \in \mathbb{R}$ and $abc(a^3 + b^3 + c^3) \geq 3$, then prove that :

$$a^2 + b^2 + c^2 \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$abc(a^3 + b^3 + c^3) \geq 3 \Rightarrow a^2b^2c^2\left(\sum_{cyc} a^3\right)^2 \geq 9$$

$$\Rightarrow \sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 \geq \frac{9}{a^2b^2c^2} \to (1)$$

$$\boxed{\text{Case 1}}\ a^2b^2c^2 \geq 1 \text{ and then : } \left(\sum_{cyc} a^2\right)^3$$

$$= \sum_{cyc} a^6 + 3\left(2a^2b^2c^2 + \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right)$$

$$= \sum_{cyc} a^6 + \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 + 6a^2b^2c^2 + 2\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right)$$

$$\overset{A-G}{\geq} \sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 + 2(3a^2b^2c^2 + 3a^2b^2c^2) \overset{\text{via (1)}}{\geq}$$

$$\frac{9}{a^2b^2c^2} + 18a^2b^2c^2 \overset{?}{\geq} 27 \Leftrightarrow 2x^4 - 3x^2 + 1 \overset{?}{\geq} 0\ (x = abc)$$

$$\Leftrightarrow (x^2 - 1)(2x^2 - 1) \overset{?}{\geq} 0 \to \text{true} \because x^2 - 1 \geq 0 \text{ and } 2x^2 \geq 2 > 1 \Rightarrow 2x^2 - 1 > 0$$

$$\therefore \left(\sum_{cyc} a^2\right)^3 \geq 27 \Rightarrow a^2 + b^2 + c^2 \geq 3$$

$$\boxed{Case\ 2}\ a^2b^2c^2 \leq 1 \text{ and then}: \left(\sum_{cyc} a^2\right)^3$$

$$= \sum_{cyc} a^6 + 3\left(2a^2b^2c^2 + \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right)$$

$$= \sum_{cyc} a^6 + \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 + 6a^2b^2c^2 + 2\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right)$$

$$\overset{A-G}{\geq} \sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 + 2\sum_{cyc}(a^4b^2 + a^4c^2) \overset{A-G}{\geq}$$

$$\sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 + 4abc(a^3 + b^3 + c^3) \overset{\substack{\text{via } (1) \\ \text{and} \\ \because abc(a^3+b^3+c^3) \geq 3}}{\geq}$$

$$\frac{9}{a^2b^2c^2} + 6a^2b^2c^2 + 12 \overset{?}{\geq} 27 \Leftrightarrow 2x^4 - 5x^2 + 3 \overset{?}{\geq} 0 \Leftrightarrow (x^2 - 1)(2x^2 - 3) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because x^2 - 1 \leq 0 \text{ and } 2x^2 \leq 2 < 3 \Rightarrow 2x^2 - 3 < 0 \Rightarrow (x^2 - 1)(2x^2 - 3) \leq 0$$

$$\therefore \left(\sum_{cyc} a^2\right)^3 \geq 27 \Rightarrow a^2 + b^2 + c^2 \geq 3 \therefore \text{combining both cases}, a^2 + b^2 + c^2 \geq 3$$

$$\forall\, a, b, c \in \mathbb{R} \,\middle|\, abc(a^3 + b^3 + c^3) \geq 3, "=" \text{ iff } a^2 = b^2 = c^2 \text{ and } a^2b^2c^2 = 1 \text{ and}$$

$$\because abc(a^3 + b^3 + c^3) \geq 3$$

$$\therefore \text{equality iff } (a = b = c = 1) \text{ or } (a = b = c = -1)\ (QED)$$

1584. **If $a, b \in \mathbb{R}$ and $a^3b^3(a^4 + b^4) \geq 2$, then prove that :**

$$a^2 + b^2 \geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^3b^3(a^4 + b^4) \geq 2 > 0 \text{ and } \because a^4 + b^4 > 0\ (a^4 + b^4 \neq 0) \therefore a^3b^3 > 0$$

$$\Rightarrow ab > 0 \rightarrow (1)$$

$$\text{Now}, \forall\, x, y \in \mathbb{R}, (x + y)^4 = (x^2 + y^2 + 2xy)^2 \geq 4(x^2 + y^2)(2xy)$$

$$(\because (m + n)^2 \geq 4mn\ \forall\, m, n \in \mathbb{R}) \therefore \text{choosing } x \equiv a^2, y \equiv b^2,$$

$$(a^2+b^2)^4 \ge 4(a^4+b^4)(2a^2b^2) \to \text{(i) and } a^2+b^2 \ge 2ab \to \text{(ii)}$$

$$\text{and} \because ab \overset{\text{via (1)}}{>} 0 \therefore \text{(i)}.\text{(ii)} \Rightarrow (a^2+b^2)^5 \ge 16a^3b^3(a^4+b^4) \overset{a^3b^3(a^4+b^4)\ge 2}{\ge} 32$$

$$\therefore a^2+b^2 \ge 2\ \forall\, a,b \in \mathbb{R} \,\big|\, a^3b^3(a^4+b^4) \ge 2,$$

$$'' ='' \text{ iff } (a=b=1) \text{ or } (a=b=-1)\ \text{(QED)}$$

1585. **If $a,b,c \in [3,5]$ and $a^2+b^2+c^2 = 50$, then prove that :**

$$a+b+c \ge 12$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$a,b,c \in [3,5] \Rightarrow 0 \le a-3, b-3, c-3 \le 2$ and assigning $x = a-3$, $y = b-3, z = c-3$ with $0 \le x,y,z \le 2$, we get :

$$\sum_{\text{cyc}}(x+3)^2 = 50 \left(\because \sum_{\text{cyc}} a^2 = 50\right) \Rightarrow \boxed{\sum_{\text{cyc}} x^2 + 6\sum_{\text{cyc}} x = 23} \to (1) \text{ and } (1) \Rightarrow$$

all of x,y,z cannot be zero and so, we now focus on the scenario when 2 among x,y,z equal zero and WLOG we may assume $y = z = 0$ and then : $(1) \Rightarrow x^2+6x-23 = 0$ whose non – negative root ≈ 2.65685; but $0 \le x \le 2$

$\therefore$ 2 among x,y,z cannot be equal to zero and hence, we now focus on the scenario when exactly one among x,y,z equals zero and WLOG we may assume $x = 0$ and then : $(1) \Rightarrow y^2+z^2+6y+6z = 23 \to \text{(i)}$

• If possible, let us assume : $y < 1$ and then : $6y + y^2 < 7\ (\because y > 0) \overset{\text{via (i)}}{\Rightarrow}$ $23 - z^2 - 6z < 7 \Rightarrow z^2+6z-16 > 0 \Rightarrow (z+8)(z-2) > 0 \to$ impossible $\because z-2 \le 0$ and $z+8 > 0 \Rightarrow (z+8)(z-2) \le 0 \therefore y \nless 1$

• If possible, let us assume : $y > 1$ and then : $y^2+z^2+6y+6z = 23 \Rightarrow$ $y^2-2y+1+z^2-4z+4+8y+10z+18 = 23 \Rightarrow 8y+10z$ $= 5-(y-1)^2-(z-2)^2 < 5 \Rightarrow 5 > 8y+10z > 8+10z \Rightarrow 10z < -3 \to$

impossible $\because z > 0 \therefore y \ngtr 1 \therefore y \nless 1, y \ngtr 1 \Rightarrow y = 1$ when $x = 0$ and putting $x = 0, y = 1$ in (i), we get : $z^2+6z-16 = 0 \Rightarrow (z+8)(z-2) = 0 \Rightarrow z = 2$ $(\because z > 0)$ and $x = 0, y = 1, z = 2 \Rightarrow x+y+z = 3 \Rightarrow a-3+b-3+c-3 = 3$ $\Rightarrow a+b+c = 12 \Rightarrow \boxed{\text{equality case for } a = 3, b = 4, c = 5 \text{ and permutations}}$

The final scenario that remains is : $0 < x,y,z \le 2$ and then :

$$(x-2)(y-2)(z-2)\le 0\Rightarrow xyz+4\sum_{cyc}x-2\sum_{cyc}xy-8\le 0$$

$$\Rightarrow -2\sum_{cyc}xy\le 8-4\sum_{cyc}x-xyz\overset{x,y,z>0\Rightarrow xyz>0}{<}8-4\sum_{cyc}x\Rightarrow\sum_{cyc}x^2+6\sum_{cyc}x$$

$$=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy+6\sum_{cyc}x<\left(\sum_{cyc}x\right)^2+8-4\sum_{cyc}x+6\sum_{cyc}x\overset{\text{via }(1)}{\Rightarrow}$$

$$23<t^2+2t+8\left(t=\sum_{cyc}x\right)\Rightarrow t^2+2t-15>0\Rightarrow(t-3)(t+5)>0$$

$$\Rightarrow t>3\left(\because t=\sum_{cyc}x>0\right)\therefore\sum_{cyc}(a-3)>3\Rightarrow a+b+c>12$$

$$\therefore a+b+c\ge 12\ \forall\, a,b,c\in[3,5]\,\middle|\,a^2+b^2+c^2=50,$$

$''=''$ iff $(a=3,b=4,c=5)$ or $(a=3,b=5,c=4)$ or $(a=4,b=5,c=3)$

or $(a=4,b=3,c=5)$ or $(a=5,b=3,c=4)$ or $(a=5,b=4,c=3)$ (QED)

1586. **If $a,b,c>0$ and $abc=1$, then prove that:**

$$\frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}$$

$$+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A,B,C>0$, $(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)$
$\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall\,x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}$$

$$+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}$$

$$\overset{abc=1}{=}\frac{\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)(b+c)}{\sqrt{bc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}+\frac{\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)(c+a)}{\sqrt{ca}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)}+\frac{\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)(a+b)}{\sqrt{ab}\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}},y=\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}},z=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}},A=a,B=b,C=c\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}ab}\overset{A-G}{\geq}$$

$$\sqrt{3}.\sqrt{3.\sqrt[3]{a^2b^2c^2}}\overset{abc=1}{=}3\therefore\frac{(\sqrt{b}+\sqrt{c})(b+c)}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+$$

$$\frac{(\sqrt{c}+\sqrt{a})(c+a)}{\sqrt{c}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{a}+\sqrt{b})}+\frac{(\sqrt{a}+\sqrt{b})(a+b)}{\sqrt{a}(\sqrt{c}+\sqrt{a})+\sqrt{b}(\sqrt{b}+\sqrt{c})}\geq 3$$

$$\forall\,a,b,c>0\,|\,abc=1,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1587. If $a,b,c>0$, *then prove that*:

$$\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)}+\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$$

$$+\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\ A,B,C>0,(A+B),(B+C),(C+A)\text{ form sides of a triangle}$$
$$(\because (A+B)+(B+C)>(C+A)\text{ and analogs})\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}\text{ form}$$
$$\text{sides of a triangle with area F (say) and }16F^2=$$
$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$
$$\text{Now},\forall\ x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$
$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$
$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$
$$\text{We have}:\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)}+\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$$
$$+\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)}$$
$$=\frac{bc}{ca+ab}\cdot\frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{bc}$$
$$+\frac{ca}{ab+bc}\cdot\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{ca}$$
$$+\frac{ab}{bc+ca}\cdot\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{ab}$$
$$=\frac{bc\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)}{ca+ab}+\frac{ca\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}{ab+bc}+\frac{ab\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}{bc+ca}$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=bc, y=ca, z=ab, A=\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}, B=\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}, C=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\left(\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}.\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}.\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\right)\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}.\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\right)}}$$

$$=\sqrt{3}.\sqrt{3.\left(\frac{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{\sqrt{a}.\sqrt{b}.\sqrt{c}}\right)^{\frac{2}{3}}} \overset{\text{Cesaro}}{\geq} \sqrt{3}.\sqrt{3.8^{\frac{2}{3}}}=6$$

$$\therefore \frac{b\sqrt{c}(\sqrt{a}+\sqrt{b})+c\sqrt{b}(\sqrt{c}+\sqrt{a})}{a(b+c)}+\frac{c\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{c}(\sqrt{a}+\sqrt{b})}{b(a+c)}$$

$$+\frac{b\sqrt{a}(\sqrt{b}+\sqrt{c})+a\sqrt{b}(\sqrt{c}+\sqrt{a})}{c(a+b)}\geq 6\ \forall a,b,c>0,''=''\text{ iff } a=b=c\ (QED)$$

1588. If $a,b,c>0$, *then prove that* :

$$\frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form

sides of a triangle with *area* F (*say*) *and* $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

Now, $\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$

Via Bergstrom, LHS of $(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\geq\frac{\sqrt{3}}{2}\to(2)$$

We have : $\frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}$

$$=\frac{c.\sqrt{\frac{b}{a}+\frac{b}{a}}.\sqrt{bc}}{b+c}+\frac{a.\sqrt{\frac{c}{b}+\frac{c}{b}}.\sqrt{ca}}{c+a}+\frac{b.\sqrt{\frac{a}{c}+\frac{a}{c}}.\sqrt{ab}}{a+b}$$

$$=\frac{\sqrt{\frac{abc}{a^2}}\left(\sqrt{c}+\frac{b}{\sqrt{a}}\right)}{b+c}+\frac{\sqrt{\frac{abc}{b^2}}\left(\sqrt{a}+\frac{c}{\sqrt{b}}\right)}{c+a}+\frac{\sqrt{\frac{abc}{c^2}}\left(\sqrt{b}+\frac{a}{\sqrt{c}}\right)}{a+b}$$

$$=\frac{c.\sqrt{ab}+b.\sqrt{bc}}{ab+ca}+\frac{a.\sqrt{bc}+c.\sqrt{ca}}{bc+ab}+\frac{b.\sqrt{ca}+a.\sqrt{ab}}{ca+bc}$$

$$=bc.\frac{\frac{c.\sqrt{ab}+b.\sqrt{bc}}{bc}}{ab+ca}+ca.\frac{\frac{a.\sqrt{bc}+c.\sqrt{ca}}{ca}}{bc+ab}+ab.\frac{\frac{b.\sqrt{ca}+a.\sqrt{ab}}{ab}}{ca+bc}$$

$$=\frac{bc\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}\right)}{ca+ab}+\frac{ca\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}\right)}{ab+bc}+\frac{ab\left(\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{b}}\right)}{bc+ca}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=bc,y=ca,z=ab,A=\sqrt{\frac{c}{a}},B=\sqrt{\frac{a}{b}},C=\sqrt{\frac{b}{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{c}{a}}.\sqrt{\frac{a}{b}}\right)}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{c}{b}}\right)}\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\sqrt{\frac{c}{b}}.\sqrt{\frac{b}{a}}.\sqrt{\frac{a}{c}}}}=3$$

$$\therefore \frac{\sqrt{\frac{c}{a}+\frac{b}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a}{b}+\frac{c}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}}+\frac{\sqrt{\frac{b}{c}+\frac{a}{c}}}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}\geq 3\ \forall a,b,c>0,''=''\text{ iff } a=b=c\ (QED)$$

1589. If $a,b,c>0$, then prove that:

$$\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

We have : $\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}$

$$+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}$$

$$=\frac{\sqrt{bc}\left(\sqrt{c(a+c)}+\sqrt{b(a+b)}\right)}{a(b+c)}+\frac{\sqrt{ca}\left(\sqrt{c(b+c)}+\sqrt{a(a+b)}\right)}{b(a+c)}$$

$$+\frac{\sqrt{ab}\left(\sqrt{b(b+c)}+\sqrt{a(a+c)}\right)}{c(a+b)}$$

$$=\frac{bc}{ca+ab}\cdot\left(\sqrt{\frac{c(a+c)}{bc}}+\sqrt{\frac{b(a+b)}{bc}}\right)+\frac{ca}{ab+bc}\cdot\left(\sqrt{\frac{c(b+c)}{ca}}+\sqrt{\frac{a(a+b)}{ca}}\right)$$

$$+\frac{ab}{bc+ca}\cdot\left(\sqrt{\frac{b(b+c)}{ab}}+\sqrt{\frac{a(a+c)}{ab}}\right)$$

$$=\frac{bc}{ca+ab}\cdot\left(\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}\right)+\frac{ca}{ab+bc}\cdot\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{a+b}{c}}\right)$$

$$+\frac{ab}{bc+ca}\cdot\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=bc,y=ca,z=ab,A=\sqrt{\frac{b+c}{a}},B=\sqrt{\frac{c+a}{b}},C=\sqrt{\frac{a+b}{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{\text{cyc}}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{\text{cyc}}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{\text{cyc}}\left(\sqrt{\frac{b+c}{a}}.\sqrt{\frac{c+a}{b}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3.3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}}\overset{\text{Cesaro}}{\geq}\sqrt{3.3.\sqrt[3]{8}}=3\sqrt{2}$$

$$\therefore\frac{\sqrt{c(a+c)}+\sqrt{b(a+b)}}{a\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+\frac{\sqrt{c(b+c)}+\sqrt{a(a+b)}}{b\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}+\frac{\sqrt{b(b+c)}+\sqrt{a(a+c)}}{c\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}$$

$$\geq 3\sqrt{2}\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1590. If $a,b,c>0$, *then prove that* :

$$\sum_{cyc}\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}}\ge 3\sqrt{a+b+c}.$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form

sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall\, x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\ge}}\frac34$$

$$\text{Via Bergstrom, LHS of }(*)\ge\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\ge}\frac34\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\ge}3xyz\sum_{cyc}x\to\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\ge}\frac{\sqrt3}{2}\to(2)$$

$$\text{We have : }\frac{\sqrt{b(b^2+bc+c^2)}+\sqrt{a(a^2+ac+c^2)}}{\sqrt{ac}+\sqrt{bc}}$$

$$+\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}}+\frac{\sqrt{a(a^2+ab+b^2)}+\sqrt{c(b^2+bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}$$

$$=\frac{\sqrt{ab}}{\sqrt{bc}+\sqrt{ca}}\cdot\left(\sqrt{\frac{b^2+bc+c^2}{a}}+\sqrt{\frac{c^2+ca+a^2}{b}}\right)$$

$$+\frac{\sqrt{bc}}{\sqrt{ca}+\sqrt{ab}}\cdot\left(\sqrt{\frac{c^2+ca+a^2}{b}}+\sqrt{\frac{a^2+ab+b^2}{c}}\right)$$

$$+\frac{\sqrt{ca}}{\sqrt{ab}+\sqrt{bc}}\cdot\left(\sqrt{\frac{a^2+ab+b^2}{c}}+\sqrt{\frac{b^2+bc+c^2}{a}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{c} x=\sqrt{ab}, y=\sqrt{bc}, z=\sqrt{ca}, \\ A=\sqrt{\frac{a^2+ab+b^2}{c}}, B=\sqrt{\frac{b^2+bc+c^2}{a}}, C=\sqrt{\frac{c^2+ca+a^2}{b}} \end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}$$

$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{a^2+ab+b^2}{c}}.\sqrt{\frac{b^2+bc+c^2}{a}}\right)}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{3ab}{c}}.\sqrt{\frac{3bc}{a}}\right)}=\sqrt{3}.\sqrt{3\sum_{cyc}b}=3.\sqrt{a+b+c}$$

$$\therefore \frac{\sqrt{b(b^2+bc+c^2)}+\sqrt{a(a^2+ac+c^2)}}{\sqrt{ac}+\sqrt{bc}}+\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}}$$

$$+\frac{\sqrt{a(a^2+ab+b^2)}+\sqrt{c(b^2+bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}\geq 3.\sqrt{a+b+c}$$

$$\forall\, a,b,c>0,''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\sqrt{a}, y:=\sqrt{b}, z:=\sqrt{c}$. We have

$$\sum_{cyc}\frac{\sqrt{b(a^2+ab+b^2)}+\sqrt{c(a^2+ac+c^2)}}{\sqrt{ab}+\sqrt{ac}} \overset{AM-GM}{\geq} \sum_{cyc}\frac{\sqrt{b.3ab}+\sqrt{c.3ac}}{\sqrt{ab}+\sqrt{ac}}$$

$$=\sqrt{3}.\sum_{cyc}\frac{y^2+z^2}{y+z} \overset{CBS}{\geq} \sqrt{3}.\frac{\left(\sum_{cyc}\sqrt{y^2+z^2}\right)^2}{2(x+y+z)}$$

$$= \sqrt{3}.\frac{2\sum_{cyc}x^2 + 2\sum_{cyc}\sqrt{(x^2+y^2)(x^2+z^2)}}{2(x+y+z)}$$

$$\overset{CBS}{\geq} \sqrt{3}.\frac{2\sum_{cyc}x^2 + 2\sum_{cyc}(x^2+yz)}{2(x+y+z)} = \sqrt{3}.\frac{3\sum_{cyc}x^2 + \left(\sum_{cyc}x\right)^2}{2(x+y+z)}$$

$$\overset{AM-GM}{\geq} 3\sqrt{x^2+y^2+z^2} = 3\sqrt{a+b+c}.$$

Equality holds iff $a = b = c$.

1591. If $a, b, c > 0$ then:

$$ab \cdot 2^{\frac{c}{b}} + bc \cdot 2^{\frac{a}{c}} + ca \cdot 2^{\frac{b}{a}} \geq 2(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$\frac{ab\left(2^{\frac{c}{b}}\right) + bc\left(2^{\frac{a}{c}}\right) + ca\left(2^{\frac{b}{a}}\right)}{ab+bc+ca} \geq \left[\left(2^{\frac{c}{b}}\right)^{ab} \cdot \left(2^{\frac{a}{c}}\right)^{bc} \cdot \left(2^{\frac{b}{a}}\right)^{ca}\right]^{\frac{1}{ab+bc+ca}} =$$

$$= \left(2^{ac} \cdot 2^{ab} \cdot 2^{bc}\right)^{\frac{1}{ab+bc+ca}} = 2^{(ab+bc+ca)\cdot\frac{1}{(ab+bc+ca)}} = 2$$

$$\therefore ab\, 2^{\frac{c}{b}} + bc \cdot 2^{\frac{a}{c}} + ca \cdot 2^{\frac{b}{a}} \geq 2(ab+bc+ca)$$

Equality holds for $a = b = c$.

Solution 2 by Khaled Abd Imouti-Syria

$$ab \cdot 2^{\frac{c}{b}} + bc \cdot 2^{\frac{a}{c}} + ca \cdot 2^{\frac{b}{a}} \overset{?}{\geq} 2(ab+bc+ca)$$

$$e_1 = \left(\frac{ab}{ab+bc+ca}\right)\left(2^{\frac{c}{b}}\right) + \left(\frac{bc}{ab+bc+ca}\right)\left(2^{\frac{a}{c}}\right) + \left(\frac{ca}{ab+bc+ca}\right)\left(2^{\frac{b}{a}}\right) \overset{?}{\geq} 2$$

By using weighted AM-GM inequality

$$e_1 = \left(\left(2^{\frac{c}{b}}\right)^{ab} \cdot \left(2^{\frac{a}{c}}\right)^{bc} \cdot \left(2^{\frac{b}{a}}\right)^{ca}\right)^{\frac{1}{ab+bc+ca}}$$

$$e_1 \geq 2^{(ca+bc+ab)\cdot\frac{1}{(ab+bc+ca)}}$$

$$e_1 \geq 2$$

Equality holds for $a=b=c$.

1592. If $a^2b>2, b^2c>2, c^2a>2$, then prove that :

$$\frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2}\ge 8$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } a^2b>2, b^2c>2, c^2a>2$$

$$\therefore \frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b-6}\overset{?}{\ge}8\Leftrightarrow\left(\sum_{cyc}a^2\right)^2+48\underset{(*)}{\overset{?}{\ge}}8\sum_{cyc}a^2b$$

$$\text{Now, } 8\sum_{cyc}a^2b\overset{\text{CBS}}{\le}8.\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)}\le 8.\sqrt{\left(\sum_{cyc}a^2\right).\frac{\left(\sum_{cyc}a^2\right)^2}{3}}$$

$$\overset{?}{\le}\left(\sum_{cyc}a^2\right)^2+48\Leftrightarrow 8.\left(\sum_{cyc}a^2\right).\sqrt{\frac{\sum_{cyc}a^2}{3}}\overset{?}{\le}\left(\sum_{cyc}a^2\right)^2+48$$

$$\Leftrightarrow 8t.3t^2\overset{?}{\le}9t^4+48\left(t=\sqrt{\frac{\sum_{cyc}a^2}{3}}\right)\Leftrightarrow 3t^4-8t^3+16\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)^2(3t^2+4t+4)\overset{?}{\ge}0\to\text{true}\because t=\sqrt{\frac{\sum_{cyc}a^2}{3}}>0\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2}\ge 8\ \forall\ a^2b>2, b^2c>2, c^2a>2,''='' \text{ iff } a=b=c$$

$$\text{and }\sqrt{\frac{\sum_{cyc}a^2}{3}}=2\Rightarrow\sum_{cyc}a^2=12\ \therefore\ ''='' \text{ iff } a=b=c=2\ (\text{QED})$$

1593. If $a>2, b>\frac{2}{3}, c>4$, then prove that :

$$\frac{a^3}{(9b-6)^2}+\frac{3b^3}{(c-4)^2}+\frac{c^3}{(3a-6)^2}\ge 6$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } a > 2, b > \frac{2}{3}, c > 4 \therefore \frac{a^3}{(9b-6)^2} + \frac{3b^3}{(c-4)^2} + \frac{c^3}{(3a-6)^2}$$

$$= \frac{a^3}{(9b-6)^2} + \frac{27b^3}{(3c-12)^2} + \frac{c^3}{(3a-6)^2} \overset{\text{Radon}}{\geq} \frac{(a+3b+c)^3}{9(a+3b+c-8)^2} = \frac{x^3}{9(x-8)^2} \overset{?}{\geq} 6$$

$$(x = a+3b+c) \Leftrightarrow x^3 - 54x^2 + 864x - 3456 \overset{?}{\geq} 0 \Leftrightarrow (x-6)(x-24)^2 \overset{?}{\geq} 0$$

$$\to \text{true} \because a + 3b + c > 2 + 3, \frac{2}{3} + 4 = 8 \Rightarrow x - 6 > 2 > 0, \text{equality iff}$$

$$\frac{a}{3b-2} = \frac{3b}{c-4} = \frac{c}{a-2} = \frac{a+3b+c}{a+3b+c-8} = \frac{24}{24-8} = \frac{3}{2} \Rightarrow \text{iff } 2a = 9b - 6 \to (\text{i}),$$

$$6b = 3c - 12 \to (\text{ii}), 2c = 3a - 6 \to (\text{iii}) \therefore (\text{i}) \text{ and } (\text{iii}) \Rightarrow 6a = 27b - 18 \text{ and}$$

$$6a = 4c + 12 \Rightarrow 27b - 18 = 4c + 12 \Rightarrow 54b = 8c + 60 \to (1) \text{ and also,}$$

$$\text{via } (\text{ii}), 54b = 27c - 108 \to (2) \therefore (1),(2) \Rightarrow 8c + 60 = 27c - 108 \Rightarrow c \overset{(*)}{=} \frac{168}{19}$$

$$\text{Putting } c = \frac{168}{19} \text{ in (ii) and (iii) respectively, we get : } b = \frac{46}{19} \text{ and } a = \frac{150}{19}$$

$$\therefore \frac{a^3}{(9b-6)^2} + \frac{3b^3}{(c-4)^2} + \frac{c^3}{(3a-6)^2} \geq 6 \ \forall \ a > 2, b > \frac{2}{3}, c > 4,$$

$$'' = '' \text{ iff } a = \frac{150}{19}, b = \frac{46}{19}, c = \frac{168}{19} \ (\text{QED})$$

1594.

$$\text{If } a, b, c > 0 \text{ and } ab^2c^3 > \frac{1}{7}, bc^2a^3 > \frac{1}{7}. ca^2b^3 > \frac{1}{7}, \text{ then prove that :}$$

$$\frac{a^7}{7ab^2c^3 - 1} + \frac{b^7}{7bc^2a^3 - 1} + \frac{c^7}{7ca^2b^3 - 1} \geq \frac{1}{2}$$

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$$\text{Since } ab^2c^3 > \frac{1}{7}, bc^2a^3 > \frac{1}{7}. ca^2b^3 > \frac{1}{7}$$

$$\therefore \frac{a^7}{7ab^2c^3 - 1} + \frac{b^7}{7bc^2a^3 - 1} + \frac{c^7}{7ca^2b^3 - 1}$$

$$= \frac{a^9}{7a^3b^2c^3 - a^2} + \frac{b^9}{7b^3c^2a^3 - b^2} + \frac{c^9}{7c^3a^2b^3 - c^2}$$

$$\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}a^3\right)^3}{21a^2b^2c^2\sum_{cyc}ab-3\sum_{cyc}a^2}\overset{?}{\geq}\frac{1}{2}$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^3\right)^3+3\sum_{cyc}a^2\underset{(*)}{\overset{?}{\geq}}21a^2b^2c^2\sum_{cyc}ab$$

$$\text{Now, Power}-\text{Mean inequality}\Rightarrow\left(\frac{\sum_{cyc}a^3}{3}\right)^{\frac{1}{3}}\geq\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{1}{2}}$$

$$\Rightarrow\sum_{cyc}a^3\geq 3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}\Rightarrow 2\left(\sum_{cyc}a^3\right)^3\geq 18\left(\frac{\sum_{cyc}a^2}{3}\right)^3.3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}$$

$$=2\left(\sum_{cyc}a^2\right)^3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}\to(1)$$

$$\text{Again, }21a^2b^2c^2\sum_{cyc}ab\overset{A-G}{\leq}\frac{7}{9}\left(\sum_{cyc}a^2\right)^4\to(2)\therefore(1),(2)\Rightarrow\text{in order}$$

$$\text{to prove }(*)\text{, it suffices to prove}: 2\left(\sum_{cyc}a^2\right)^3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}+3\sum_{cyc}a^2\geq\frac{7}{9}\left(\sum_{cyc}a^2\right)^4$$

$$\Leftrightarrow 2.27t^6.t^3+9t^2\geq\frac{7}{9}.81t^8\left(t=\sqrt{\frac{\sum_{cyc}a^2}{3}}\right)\Leftrightarrow t^2(6t^7-7t^6+1)\geq 0$$

$$\Leftrightarrow(t-1)^2(6t^5+5t^4+4t^3+3t^2+2t+1)\geq 0\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a^7}{7ab^2c^3-1}+\frac{b^7}{7bc^2a^3-1}+\frac{c^7}{7ca^2b^3-1}\geq\frac{1}{2}\ \forall a,b,c>0\,|\,ab^2c^3>\frac{1}{7},$$

$$bc^2a^3>\frac{1}{7}.ca^2b^3>\frac{1}{7},''=''\text{ iff }a=b=c\text{ and }\sqrt{\frac{\sum_{cyc}a^2}{3}}=1\Rightarrow\sum_{cyc}a^2=3$$

$\therefore$ $''=''$ iff $a=b=c=1$ (QED)

1595. If $a>\frac{1}{2}$, $b>\frac{1}{4}$, $c\frac{1}{3}$, prove the inequality

$$\frac{3a^2}{4b-1}+\frac{2b^2}{9c-3}+\frac{c^2}{2a-1}\geq 2.\text{ In what case is equality possible?}$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x \coloneqq 3a+2b+3c$. By CBS inequality, we have

$$\frac{3a^2}{4b-1}+\frac{2b^2}{9c-3}+\frac{c^2}{2a-1}\geq\frac{(3a+2b+3c)^2}{3(4b-1)+2(9c-3)+9(2a-1)}=$$

$$=\frac{x^2}{6(x-3)}=2+\frac{(x-6)^2}{6(x-3)}\geq 2$$

Equality holds when $3a+2b+3c=x=6$ and

$$\frac{a}{4b-1}=\frac{b}{9c-3}=\frac{c}{3(2a-1)}=\frac{3.a+2.b+3.c}{3.(4b-1)+2.(9c-3)+3.3(2a-1)}=$$

$$=\frac{x}{6x-18}=\frac{1}{3} \text{ then } a=\frac{17}{21}, b=\frac{6}{7}, c=\frac{13}{21}$$

1596. If $a,b,c>0$ such that $ab,bc,ca>1$, then

$$a^3(bc-1)+b^3(ca-1)+c^3(ab-1)\leq\frac{4}{9}a^3b^3c^3$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : for $x>0$, *we have* $x-1\leq\frac{4}{27}x^3$.

Proof: By AM − GM inequality, we have

$$\frac{4}{27}x^3+1=\frac{4}{27}x^3+\frac{1}{2}+\frac{1}{2}\geq 3\sqrt[3]{\frac{4}{27}x^3.\frac{1}{2}.\frac{1}{2}}=x,$$

with equality for $\frac{4}{27}x^3=\frac{1}{2}$ or $x=\frac{3}{2}$. Using this lemma, we have

$$a^3(bc-1)+b^3(ca-1)+c^3(ab-1)\leq$$

$$\leq a^3.\frac{4}{27}(bc)^3+b^3.\frac{4}{27}(ca)^3+c^3.\frac{4}{27}(ab)^3=\frac{4}{9}a^3b^3c^3.$$

Equality holds for $a=b=c=\sqrt{\frac{3}{2}}$.

1597. If $a, b, c > 0$ such that : $a+b+c=3$ and $\lambda \geq 0$, then :

$$\frac{a}{b}(b+\lambda c)+\frac{b}{c}(c+\lambda a)+\frac{c}{a}(a+\lambda b) \geq \frac{3}{2}(1+(2\lambda+1)abc)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}(b+\lambda c)+\frac{b}{c}(c+\lambda a)+\frac{c}{a}(a+\lambda b) \geq \frac{3}{2}(1+(2\lambda+1)abc)$$

$$\Leftrightarrow \sum_{cyc} a+\lambda \sum_{cyc} \frac{ca}{b} \geq \frac{3}{2}+\frac{3}{2}(2\lambda+1)abc$$

$$\overset{a+b+c=3}{\Leftrightarrow} \frac{\left(3-\frac{3}{2}\right)}{27}\left(\sum_{cyc} a\right)^3+\frac{\lambda}{9abc}\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \geq 3\lambda abc+\frac{3}{2}abc$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)^3+\frac{2\lambda}{abc}\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \geq 54\lambda abc+27abc \to \text{true}$$

$$\because \left(\sum_{cyc} a\right)^3 \overset{A-G}{\geq} 27abc \text{ and } \because 2\lambda\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \overset{\substack{A-G \\ \text{and} \\ \because \lambda \geq 0}}{\geq}$$

$$2\lambda\left(3\sqrt[3]{a^4b^4c^4}\right)\left(3\sqrt[3]{abc}\right)^2 = 54\lambda a^2b^2c^2 \Rightarrow \frac{2\lambda}{abc}\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \geq 54\lambda abc$$

$$\therefore \frac{a}{b}(b+\lambda c)+\frac{b}{c}(c+\lambda a)+\frac{c}{a}(a+\lambda b) \geq \frac{3}{2}(1+(2\lambda+1)abc)$$

$$\forall a, b, c > 0 \mid a+b+c=3, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$

1598. If $x, y, z, \lambda > 0, x+y+z=3$ then:

$$1)\ \frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3} \geq \frac{3}{(\lambda+1)^3}$$

$$2)\ \frac{x}{(y+\lambda)^3}+\frac{y}{(z+\lambda)^3}+\frac{z}{(x+\lambda)^3} \geq \frac{3}{(\lambda+1)^3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$1)\ \frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3} \geq \frac{3}{(\lambda+1)^3}$$

$$Let\ f(p)=\frac{1}{(p+\lambda)^3},p\in(0,3) and\ \lambda\geq 0,f'(p)=-\frac{3}{(p+\lambda)^4},$$

$$f''(p)=\frac{12}{(p+\lambda)^5}>0\ so\ f\ is\ convex\in(0,3). Using\ Jensen\ inequality$$

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3f(1)\ or$$

$$\frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3}\geq\frac{3}{(1+\lambda)^3}, equality\ holds\ a=b=c$$

$$2)\ \frac{x}{(y+\lambda)^3}+\frac{y}{(z+\lambda)^3}+\frac{z}{(x+\lambda)^3}\geq\frac{3}{(\lambda+1)^3}$$

$$LHS=\sum\frac{x^4}{(xy+\lambda x)^3}\overset{Radon}{\geq}\frac{(x+y+z)^4}{\left(xy+yz+zx+\lambda(x+y+z)\right)^3}$$

$$\geq\frac{(x+y+z)^4}{\left(\frac{(\sum x)^2}{3}+\lambda(x+y+z)\right)^3}=\frac{3}{(\lambda+1)^3}(as\ x+y+z=3)$$

1599. If $a,b,c>0, a+b+c=1$ then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{27}{4}(ab+bc+ca)\geq\frac{45}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{27}{4}(ab+bc+ca)=$$

$$=\frac{3}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{27}{4}(ab+bc+ca)\overset{Bergstrom}{\geq}$$

$$\geq\frac{3}{4}\frac{(1+1+1)^2}{a+b+c}+\frac{1}{4}\frac{ab+bc+ca}{abc}+\frac{27}{4}(ab+bc+ca)\overset{AM-GM}{\geq}$$

$$\geq \frac{3}{4}.9+2\sqrt{\frac{27}{4}.\frac{1}{4}.\frac{1}{abc}.(ab+bc+ca)^2}\ (since\ a+b+c=1)\overset{(\sum x)^2\geq 3\sum xy}{\geq}$$

$$\geq \frac{27}{4}+\frac{1}{2}\sqrt{\frac{27}{abc}.3abc(a+b+c)}=\frac{27}{4}+\frac{9}{2}=\frac{45}{4}$$

Equality holds for:

$$a=b=c=\frac{1}{3}$$

1600. **If $a,b,c>0$ such that : $abc=1$ and $0<\lambda\leq 27n$, then :**

$$n(a+b+c)(ab+bc+ca)+\frac{\lambda}{ab+bc+ca}\geq 9n+\frac{\lambda}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a\overset{A-G}{\geq} 3\sqrt[3]{abc}=3\rightarrow (1)$$

$$\therefore n(a+b+c)(ab+bc+ca)-9n\geq$$

$$\geq 3n\sum_{cyc}ab-9n=3n\left(\sum_{cyc}ab-3\right)\overset{?}{\geq}\frac{\lambda}{3}-\frac{\lambda}{ab+bc+ca}$$

$$\Leftrightarrow 3n\left(\sum_{cyc}ab-3\right)\underset{(*)}{\overset{?}{\geq}}\lambda\left(\frac{\sum_{cyc}ab-3}{3\sum_{cyc}ab}\right)$$

$$\text{Now}, 0<\lambda\leq 27n\Rightarrow RHS\ of\ (*)\leq 27n\left(\frac{\sum_{cyc}ab-3}{3\sum_{cyc}ab}\right)$$

$$\left(\because \sum_{cyc}ab\overset{A-G}{\geq}3\sqrt[3]{a^2b^2c^2}\overset{abc=1}{=}3\Rightarrow\sum_{cyc}ab-3\geq 0\right)\overset{?}{\leq}3n\left(\sum_{cyc}ab-3\right)$$

$$\Leftrightarrow\sum_{cyc}ab-3-3\left(\frac{\sum_{cyc}ab-3}{\sum_{cyc}ab}\right)\overset{?}{\geq}0\ (\because n>0)\Leftrightarrow\frac{1}{\sum_{cyc}ab}\left(\sum_{cyc}ab-3\right)^2\overset{?}{\geq}0$$

$$\rightarrow \textbf{true} \Rightarrow (*) \textbf{ is true} \therefore \mathbf{n}(a+\mathbf{b}+\mathbf{c})(a\mathbf{b}+\mathbf{bc}+\mathbf{c}a) - 9\mathbf{n} \geq \frac{\lambda}{3} - \frac{\lambda}{a\mathbf{b}+\mathbf{bc}+\mathbf{c}a}$$

$$\Rightarrow \mathbf{n}(a+\mathbf{b}+\mathbf{c})(a\mathbf{b}+\mathbf{bc}+\mathbf{c}a) + \frac{\lambda}{a\mathbf{b}+\mathbf{bc}+\mathbf{c}a} \geq 9\mathbf{n} + \frac{\lambda}{3}$$

$$\forall\, a, \mathbf{b}, \mathbf{c} > 0 \,\Big|\, a\mathbf{bc} = 1 \textbf{ and } 0 < \lambda \leq 27n, '' ='' \textbf{ iff } a = \mathbf{b} = \mathbf{c} = 1 \ (\textbf{QED})$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru