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S.2352 If $m \geq 0$ and $x, y, z > 0$ then in ΔABC holds:

$$\left(\frac{x}{\sqrt{yz}}\right)^m \frac{a^{m+2}}{h_a^m} + \left(\frac{y}{\sqrt{zx}}\right)^m \frac{b^{m+2}}{h_b^m} + \left(\frac{z}{\sqrt{xy}}\right)^m \frac{c^{m+2}}{h_c^m} \geq 2^{m+2}\left(\sqrt{3}\right)^{1-m} F$$

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We have $ah_a = bh_b = ch_c = 2F$.

Using $AM - GM$ inequality and Carlitz inequality $(abc)^{2/3} \geq \frac{4}{3}\sqrt{3}F$, it follows that

$$\left(\frac{x}{\sqrt{yz}}\right)^m \frac{a^{m+2}}{h_a^m} + \left(\frac{y}{\sqrt{zx}}\right)^m \frac{b^{m+2}}{h_b^m} + \left(\frac{z}{\sqrt{xy}}\right)^m \frac{c^{m+2}}{h_c^m} =$$

$$= \left(\frac{x}{\sqrt{yz}}\right)^m \frac{a^{2m+2}}{(2F)^m} + \left(\frac{y}{\sqrt{zx}}\right)^m \frac{b^{2m+2}}{(2F)^m} + \left(\frac{z}{\sqrt{xy}}\right)^m \frac{c^{2m+2}}{(2F)^m} \overset{AM-GM}{\geq}$$

$$\geq \frac{3}{(2F)^m}\left(\left(\frac{x}{\sqrt{yz}} \cdot \frac{y}{\sqrt{zx}} \cdot \frac{z}{\sqrt{xy}}\right)^m a^{2m+2} b^{2m+2} c^{2m+2}\right)^{\frac{1}{3}} = \frac{3}{(2F)^m}\left((abc)^{\frac{2}{3}}\right)^{m+1} \geq$$

$$\overset{CARLITZ}{\geq} \frac{3}{(2F)^m}\left(\frac{4}{3}\sqrt{3}F\right)^{m+1} = 2^{m+2}\left(\sqrt{3}\right)^{1-m} F.$$

Equality holds if and only if ΔABC is equilateral.