ROMANIAN MATHEMATICAL MAGAZINE

S.2353 If $m \geq 0$ and $x, y, z > 0$, then in ΔABC holds:

$$\frac{x^{m+1} \cdot a^{2m+1}}{(y+z)^{m+1} \cdot h_a} + \frac{y^{m+1} \cdot b^{2m+1}}{(z+x)^{m+1} \cdot h_b} + \frac{z^{m+1} \cdot c^{2m+1}}{(x+y)^{m+1} \cdot h_c} \geq 2^m \left(\sqrt{3}\right)^{1-m} \cdot F^m$$

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We have $ah_a = bh_b = ch_c = 2F$. By Power Mean inequality we have

$$\left(\frac{x^{m+1} + y^{m+1} + z^{m+1}}{3}\right)^{\frac{1}{m+1}} \geq \frac{x+y+z}{3} \Leftrightarrow$$

$$3^m\left(x^{m+1} + y^{m+1} + z^{m+1}\right) \geq (x+y+z)^{m+1} \quad (1)$$

Applying (1), Tsintsifas' inequality:

$$\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2 \geq 2\sqrt{3} \cdot F$$

and $AM - GM$ inequality, it follows that

$$\frac{x^{m+1} \cdot a^{2m+1}}{(y+z)^{m+1} \cdot h_a} + \frac{y^{m+1} \cdot b^{2m+1}}{(z+x)^{m+1} \cdot h_b} + \frac{z^{m+1} \cdot c^{2m+1}}{(x+y)^{m+1} \cdot h_c} =$$

$$= \frac{x^{m+1} \cdot a^{2m+2}}{(y+z)^{m+1} \cdot ah_a} + \frac{y^{m+1} \cdot b^{2m+2}}{(z+x)^{m+1} \cdot bh_b} + \frac{z^{m+1} \cdot c^{2m+2}}{(x+y)^{m+1} \cdot ch_c} =$$

$$= \frac{1}{2F}\left(\left(\frac{xa^2}{y+z}\right)^{m+1} + \left(\frac{yb^2}{z+x}\right)^{m+1} + \left(\frac{zc^2}{x+y}\right)^{m+1}\right) \geq$$

$$\geq \frac{1}{2 \cdot 3^m \cdot F}\left(\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2\right)^{m+1} \geq$$

$$\geq \frac{1}{2 \cdot 3^m \cdot F}\left(2\sqrt{3}F\right)^{m+1} = 2^m\left(\sqrt{3}\right)^{1-m} \cdot F^m.$$

Equality holds if and only if ΔABC is equilateral and $x = y = z$.