ROMANIAN MATHEMATICAL MAGAZINE

S.2354 If $x,y,z\geq 0, u,v\in R$ then in triangles $A_kB_kC_k, k=1,2,3$ holds:

$$\frac{m_{a_1}^u\cdot a_1^{x+u}\cdot b_2^{y-v}\cdot c_3^z}{h_{b_2}^v}+\frac{m_{b_1}^u\cdot b_1^{x+u}\cdot c_2^{y-v}\cdot a_3^z}{h_{c_2}^v}+\frac{m_{c_1}^u\cdot c_1^{x+u}\cdot a_2^{y-v}\cdot b_3^z}{h_{a_2}^v}\geq$$

$$\geq 2^{x+y+z+u-v}\cdot 3^{1-\frac{x+y+z}{4}}\cdot\left(\sqrt{F_1}\right)^{x+2u}\cdot\left(\sqrt{F_2}\right)^{y-2v}\cdot\left(\sqrt{F_3}\right)^z$$

Proposed by D.M.Bătinețu-Giurgiu, Gilena Dobrică – Romania

Solution by Titu Zvonaru-Romania

We have

$$m_{a_1}\geq h_{a_1}, m_{b_1}\geq h_{b_1}, m_{c_1}\geq h_{c_1} \text{ and } a_1h_{a_1}=b_1h_{b_1}=c_1h_{c_1}=2F_1,$$

$$a_2h_{a_2}=b_2h_{b_2}=c_2h_{c_2}=2F_2.$$

Applying $AM-GM$ inequality and Carlitz's inequality $(abc)^{2/3}\geq\frac{4}{\sqrt{3}}F$

(for the triangles $A_kB_kC_k, k=1,2,3$), it follows that

$$\frac{m_{a_1}^u\cdot a_1^{x+u}\cdot b_2^{y-v}\cdot c_3^z}{h_{b_2}^v}+\frac{m_{b_1}^u\cdot b_1^{x+u}\cdot c_2^{y-v}\cdot a_3^z}{h_{c_2}^v}+\frac{m_{c_1}^u\cdot c_1^{x+u}\cdot a_2^{y-v}\cdot b_3^z}{h_{a_2}^v}\geq$$

$$\geq\frac{\left(a_1h_{a_1}\right)^u\cdot a_1^x\cdot b_2^y\cdot c_3^z}{\left(b_2h_{b_2}\right)^v}+\frac{\left(b_1h_{b_1}\right)^u\cdot b_1^x\cdot c_2^y\cdot a_3^z}{\left(c_2h_{c_2}\right)^v}+\frac{\left(c_1h_{c_1}\right)^u\cdot c_1^x\cdot a_2^y\cdot b_3^z}{\left(a_2h_{a_2}\right)^v}=$$

$$=\frac{2^uF_1^u\cdot a_1^x\cdot b_2^y\cdot c_3^z}{2^vF_2^v}+\frac{2^uF_1^u\cdot b_1^x\cdot c_2^y\cdot a_3^z}{2^vF_2^v}+\frac{2^uF_1^u\cdot c_1^x\cdot a_2^y\cdot b_3^z}{2^vF_2^v}=$$

$$=\frac{2^{u-v}F_1^u}{F_2^v}\left(a_1^x\cdot b_2^y\cdot c_3^z+b_1^x\cdot c_2^y\cdot a_3^z+c_1^x\cdot a_2^y\cdot b_3^z\right)\geq$$

$$=\frac{3\cdot 2^{u-v}F_1^u}{F_2^v}\left((a_1b_1c_1)^x\cdot(a_2b_2c_2)^y\cdot(a_3b_3c_3)^z\right)^{\frac{1}{3}}\geq$$

$$\geq\frac{3\cdot 2^{u-v}F_1^u}{F_2^v}\left(\left(\frac{4}{\sqrt{3}}F_1\right)^{\frac{x}{2}}\cdot\left(\frac{4}{\sqrt{3}}F_2\right)^{\frac{y}{2}}\cdot\left(\frac{4}{\sqrt{3}}F_3\right)^{\frac{z}{2}}\right)^{\frac{1}{3}}$$

$$=2^{x+y+z+u-v}\cdot 3^{1-\frac{x+y+z}{4}}\cdot\left(\sqrt{F_1}\right)^{x+2u}\cdot\left(\sqrt{F_2}\right)^{y-2v}\cdot\left(\sqrt{F_3}\right)^z.$$

Equality holds if and only if triangles $A_kB_kC_k, k=1,2,3$ are equilateral and congruent, and $u=v$.