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S.2356 If $m,n>0, m\neq n$ and in ΔABC, $A_1,A_2\in(BC)$, $B_1,B_2\in(CA)$, $C_1,C_2\in(AB)$ such that $BA_1=mA_1C, BA_2=nA_2C, CB_1=mB_1A$, $CB_2=nB_2A, AC_1=mC_1B, AC_2=nC_2B$ then if $a_1=B_1C_1, b_1=C_1A_1$, $c_1=A_1B_1$ and $a_2=B_2C_2, b_2=C_2A_2, c_2=A_2B_2$ holds

$$a_1a_2+b_1b_2+c_1c_2\geq\frac{4\sqrt{3}\cdot\sqrt{mn}}{(m+1)(n+1)}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Dorina Goiceanu – Romania

Solution by Titu Zvonaru-Romania

By $BA_1=mA_1C, BA_1+A_1C=a$, we get $BA_1=\frac{am}{m+1}, A_1C=\frac{a}{m+1}$, and similar $CB_1=\frac{bm}{m+1}, B_1A=\frac{b}{m+1}, AC_1=\frac{cm}{m+1}, C_1B=\frac{c}{m+1}$.

We obtain $[AB_1C_1]=\frac{AB_1AC_1\sin A}{2}=\frac{bcm\sin A}{2(m+1)^2}=\frac{m}{(m+1)^2}\cdot F$. It follows that

$$[A_1B_1C_1]=[ABC]-[AB_1C_1]-[BC_1A_1]-[CA_1B_1]=$$

$$=F-\frac{3}{(m+1)^2}\cdot F=\frac{m^2-m+1}{(m+1)^2}\cdot F.$$

By $AM-GM$ we have $m^2-m+1=m^2+1-m\geq 2m-m=m$. It results that

$$[A_1B_1C_1]=\frac{m^2-m+1}{(m+1)^2}\cdot F\geq\frac{m}{(m+1)^2}\cdot F\quad (1)$$

Analog we obtain

$$[A_2B_2C_2]\geq\frac{n}{(n+1)^2}\cdot F\quad (2)$$

Using $AM-GM$ inequality, Carlitz inequality $(abc)^{2/3}\geq\frac{4}{3}\sqrt{3}F$ (for the triangles $A_1B_1C_1$ and $A_2B_2C_2$), the inequalities (1) and (2), it follows that

$$a_1a_2+b_1b_2+c_1c_2\geq 3(a_1b_1c_1)^{\frac{1}{3}}(a_2b_2c_2)^{\frac{1}{3}}\geq$$

$$\geq 3\sqrt{\frac{4}{3}\sqrt{3}\frac{m}{(m+1)^2}\cdot F\cdot\frac{4}{3}\sqrt{3}\frac{n}{(n+1)^2}\cdot F}=\frac{4\sqrt{3}\cdot\sqrt{mn}}{(m+1)(n+1)}\cdot F.$$

Since $m\neq n$, inequality is strict. If we accept $m=n$, then the equality holds if and only if $m=n=1$ and the triangle ABC is equilateral.