ROMANIAN MATHEMATICAL MAGAZINE

S.2358 In ΔABC,if $a,b,c>0$ and $a,b,c\le 1$ then holds:

$$\sum_{cyc}\sqrt{(a^4+a^2+1)(b^4+b^2+1)}\ge 12\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Ileana Duma - Romania

Solution by Titu Zvonaru-Romania

By $AM-GM$ inequality we have

$$a^4+a^2+1=a^4+1+a^2\ge 2\sqrt{a^4\cdot 1}+a^2=2a^2+a^2=3a^2 \quad (1)$$

Using Gordon's inequality $ab+bc+ca\ge 4\sqrt{3}F$, we obtain:

$$\sum_{cyc}\sqrt{(a^4+a^2+1)(b^4+b^2+1)}\overset{(1)}{\ge}\sum_{cyc}\sqrt{3a^2\cdot 3b^2}=$$

$$=\sqrt{9a^2b^2}+\sqrt{9b^2c^2}+\sqrt{9c^2a^2}=3(ab+bc+ca)\overset{GORDON}{\ge}12\sqrt{3}\cdot F.$$

Equality holds if and only if $a=b=c=1$.