ROMANIAN MATHEMATICAL MAGAZINE

S.2359 If $x, y, z > 0$ then in ΔABC holds:

$$(x+y)ab+(y+z)bc+(z+x)ca \geq 8\sqrt{xy+yz+zx}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu, Maria Lavinia Popa – Romania

Solution by Titu Zvonaru-Romania

First, we will prove that

$$27(x+y)^2(y+z)^2(z+x)^2 \geq 64(xy+yz+zx)^3 \quad (1)$$

First we prove that $9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx) \quad (2)$

which is equivalent with:

$$x(z-y)^2+y(x-z)^2+z(y-x)^2 \geq 0$$

and $(x+y+z)^2 \geq 3(xy+yz+zx) \quad (3)$

we obtain by squaring (2)**:**

$$81(x+y)^2(y+z)^2(z+x)^2 \geq 64(x+y+z)^2(xy+yz+zx)^2 \overset{(3)}{\geq} 192(xy+yz+zx)^3,$$

hence the inequality (1) **is true.**

Using (1), $AM-GM$ **and Carlitz' inequality** $(abc)^{2/3} \geq \frac{4}{3}\sqrt{3}F$, **we obtain:**

$$(x+y)ab+(y+z)bc+(z+x)ca \overset{AM-GM}{\geq}$$

$$\geq 3\left((x+y)(y+z)(z+x)(abc)^2\right)^{1/3} \overset{CARLITZ}{\geq} \geq 8\sqrt{xy+yz+zx}\cdot F.$$

Equality holds if and only if $x=y=z$ **and** ΔABC **is equilateral.**