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S.2363 If $m>0$ and $M\in Int(\Delta ABC)$, $d_a=d(M,BC)$, $d_b=d(M,CA)$, $d_c=d(M,AB)$ then

$$\frac{(ab)^{m+2}}{(d_ad_b)^m}+\frac{(bc)^{m+2}}{(d_bd_c)^m}+\frac{(ca)^{m+2}}{(d_cd_a)^m}\geq 3\cdot 4^{m+2}F^2$$

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Solution by Titu Zvonaru-Romania

We have $ad_a+bd_b+cd_c=2F$.

By $AM-GM$ inequality we obtain:

$$\left((ad_a)(bd_b)(cd_c)\right)\leq\frac{(ad_a+bd_b+cd_c)^3}{27}=\frac{8F^3}{27}$$

Using $AM-GM$ inequality and Carliz inequality $(abc)^2\geq\frac{4\sqrt{3}}{3}F$, we obtain:

$$\frac{(ab)^{m+2}}{(d_ad_b)^m}+\frac{(bc)^{m+2}}{(d_bd_c)^m}+\frac{(ca)^{m+2}}{(d_cd_a)^m}=\frac{(ab)^{2m+2}}{(ad_abd_b)^m}+\frac{(bc)^{2m+2}}{(bd_bcd_c)^m}+\frac{(ca)^{2m+2}}{(cd_cad_a)^m}\overset{AM-GM}{\geq}$$

$$\geq\frac{3(abc)^{\frac{4m+4}{3}}}{(ad_abd_bcd_c)^{\frac{2m}{3}}}\overset{CARLITZ}{\geq}\frac{3\left(\frac{4}{\sqrt{3}}\right)^{2m+2}F^{2m+2}}{\left(\frac{2F}{3}\right)^{2m}}=3\cdot 4^{m+2}F^2$$

Equality holds if and only if ΔABC is equilateral and M is circumcenter.