ROMANIAN MATHEMATICAL MAGAZINE

S.2364 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x^2+1}{y+z}\cdot a^4+\frac{y^2+1}{z+x}\cdot b^4+\frac{z^2+1}{x+y}\cdot c^4\ge 16F^2$$

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$$2(x^2+1)(y^2+1)(z^2+1)-2(x+y)(y+z)(z+x)=$$

$$=2(xyz-1)^2+(x^2+y^2)(z-1)^2+(y^2+z^2)(x-1)^2+(z^2+x^2)(y-1)^2\ge 0,$$

$$2(x^2+1)(y^2+1)(z^2+1)-2(x+y)(y+z)(z+x)\ge 0$$

$$(x^2+1)(y^2+1)(z^2+1)\ge (x+y)(y+z)(z+x)\quad (1)$$

Applying $AM-GM$ inequality and Carlitz' inequality $(abc)^{2/3}\ge \frac{4}{3}\sqrt{3}F$,we obtain

$$\frac{x^2+1}{y+z}\cdot a^4+\frac{y^2+1}{z+x}\cdot b^4+\frac{z^2+1}{x+y}\cdot c^4\ge 3\left(\frac{x^2+1}{y+z}\cdot\frac{y^2+1}{z+x}\cdot\frac{z^2+1}{x+y}\cdot a^4b^4c^4\right)^{\frac{1}{3}}\overset{(1)}{\ge}$$

$$\ge 3(abc)^{\frac{4}{3}}\ge 3\cdot\frac{16}{3}F^2=16F^2.$$

Equality holds if and only if $x=y=z=1$ and ΔABC s equilateral.