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S.2366 If $t,x,y,z>0$, then:

$$\left(\left(\frac{x}{y+z}+\frac{y}{z+x}\right)^2+t^2\right)\left(\left(\frac{y}{z+x}+\frac{z}{x+y}\right)^2+t^2\right)\left(\left(\frac{z}{x+y}+\frac{x}{y+z}\right)^2+t^2\right)\geq\frac{27}{4}t^4$$

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It is known the inequality of Arkady Alt (see for example $[1]$):

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2 \qquad (1)$$

with equality if and only if $x=y=z, t=x\sqrt{2}$.

Using (1) and Nesbitt inequality $\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$, it follows that

$$\left(\left(\frac{x}{y+z}+\frac{y}{z+x}\right)^2+t^2\right)\left(\left(\frac{y}{z+x}+\frac{z}{x+y}\right)^2+t^2\right)\left(\left(\frac{z}{x+y}+\frac{x}{y+z}\right)^2+t^2\right)\geq$$

$$\geq\frac{3}{4}t^4\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{y}{z+x}+\frac{z}{x+y}+\frac{z}{x+y}+\frac{x}{y+z}\right)^2\geq$$

$$\geq\frac{3}{4}t^4\left(\frac{3}{2}+\frac{3}{2}\right)^2=\frac{27}{4}t^4.$$

Equality holds if and only if $x=y=z, t=\sqrt{2}$.

$[1]$ D.M.Bătinețu-Giurgiu, N. Papacu, I. Tudor, *Asupra unei inegalități propusă APMO 2004*, Recreații Matematice nr. 1/2024

ARKADI ALT'S INEQUALITY

If $t,x,y,z>0$ then the following relationship holds:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}t^4(x+y+z)^2$$

with equality if and only if $x=y=z=\frac{t}{\sqrt{2}}$.

Proof: We have

$$(x^2+t^2)(y^2+t^2)\geq\frac{3}{4}t^2((x+y)^2+t^2)\Leftrightarrow\left(xy-\frac{t^2}{2}\right)^2+\frac{t^2}{4}(x-y)^2\geq 0.$$

Applying Cauchy-Buniakovski-Schwarz inequality we obtain

$$(x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3t^2}{4}\left((x+y)^2+t^2\right)(t^2+z^2) \geq$$

$$\geq \frac{3t^2}{4}\left(t(x+y)+tz\right)^2 = \frac{3}{4}t^4(x+y+z)^2.$$

The equality holds if and only if $x=y=z=\frac{t}{\sqrt{2}}$.