ROMANIAN MATHEMATICAL MAGAZINE

S.2380 If $a,b,c>0$, then:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{2^{\frac{a+b+c}{3}}}$$

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We denote $x^3=2^a, y=2^b, z^3=2^c$. We have to prove that

$$\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\geq\frac{x^3+y^3+z^3}{xyz}\quad (1)$$

Applying $AM-GM$ inequality it follows that

$$\frac{x^3}{y^3}+\frac{x^3}{y^3}+\frac{y^3}{z^3}\geq 3\left(\frac{x^3}{y^3}\cdot\frac{x^3}{y^3}\cdot\frac{y^3}{z^3}\right)^{\frac{1}{3}}=\frac{3x^2}{yz}\quad (2)$$

$$\frac{y^3}{z^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\geq 3\left(\frac{y^3}{z^3}\cdot\frac{y^3}{z^3}\cdot\frac{z^3}{x^3}\right)^{\frac{1}{3}}=\frac{3y^2}{zx}\quad (3)$$

$$\frac{z^3}{x^3}+\frac{z^3}{x^3}+\frac{x^3}{y^3}\geq 3\left(\frac{z^3}{x^3}\cdot\frac{z^3}{x^3}\cdot\frac{x^3}{y^3}\right)^{\frac{1}{3}}=\frac{3z^2}{yz}\quad (4)$$

Adding inequalities $(2),(3),(4)$ we obtain the inequality (1).

Equality holds if and only if $x=y=z$, that is $a=b=c$.