ROMANIAN MATHEMATICAL MAGAZINE

S.2381 If $a \in (0, \pi/2)$ then

$$\frac{\sin 2a}{\sin a + \cos a} + \frac{1}{\sqrt{2}} \ge \sqrt{\frac{\sin 2a}{2}} + \frac{\sin a + \cos a}{2}$$

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Denoting $x = \sin a, y = \cos a$, we have $x, y > 0, x^2 + y^2 = 1$. We have to prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \ge \sqrt{xy} + \frac{x+y}{2}.$$

This inequality is equivalent to

$$\sqrt{\frac{x^2+y^2}{2}} - \sqrt{xy} \ge \frac{x+y}{2} - \frac{2xy}{x+y} \Leftrightarrow \frac{\frac{x^2+y^2}{2} - xy}{\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}} \ge \frac{(x-y)^2}{2(x+y)}$$

$$(x-y)^2 \ge \frac{(x-y)^2}{x+y}\left(\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}\right).$$

If $x = y$ then we have equality, else it remains to prove that

$$\Leftrightarrow x + y \ge \sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}.$$

The last inequality is true because

$$(x+y)^2 \ge \frac{x^2+y^2}{2} + xy + \frac{x^2+y^2}{2} + xy \ge \frac{x^2+y^2}{2} + xy + 2\sqrt{\frac{(x^2+y^2)xy}{2}}$$
$$= \left(\sqrt{\frac{x^2+y^2}{2}} + \sqrt{xy}\right)^2.$$

Equality holds if and only if $x = y$ that is if and only if $a = \pi/2$.