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S.2384 $ABCD$ – convex quadrilateral, $AB = a, BC = b, CD = c,$

$DA = d, AC = e, BD = f, AC \perp BD$. Prove that:

$$\frac{a^3b}{c^2} + \frac{b^3c}{d^2} + \frac{c^3d}{a^2} + \frac{d^3a}{b^2} \geq \frac{8e^2f^2}{(e+f)^2}$$

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Since $AC \perp BD$, we have $a^2 + c^2 = b^2 + d^2 \quad (1)$, and by Ptolemy inequality we have $ac + bd \geq ef \quad (2)$.

By $AM - GM$ inequality, using (2), we obtain

$$\frac{8e^2f^2}{(e+f)^2} \leq \frac{8e^2f^2}{4ef} = 2ef \leq 2(ac+bd) \quad (3)$$

Applying Radon's inequality, yields that:

$$\frac{a^3b}{c^2} + \frac{b^3c}{d^2} + \frac{c^3d}{a^2} + \frac{d^3a}{b^2} = \frac{(ab)^3}{(bc)^2} + \frac{(bc)^3}{(cd)^2} + \frac{(cd)^3}{(da)^2} + \frac{(da)^3}{(ab)^2} \geq \frac{(ab+bc+cd+da)^3}{(ab+bc+cd+da)^2}$$
$$= ab + bc + cd + da \quad (4)$$

Using inequalities (3) and (4), it results that it suffices to prove that

$$(a+c)(b+d) \geq 2ac + 2bd \qquad (5)$$

By the relation (1), the relation (5) is equivalent to

$$(a+c)(b+d) + a^2 + c^2 \geq b^2 + d^2 + 2ac + 2bd$$

$$(a+c)(b+d) - (b+d)^2 + (a-c)^2 \geq 0 \Leftrightarrow (b+d)(a+c-b-d) + (a-c)^2 \geq 0 \qquad (6)$$

In a similar way, the inequality (5) is equivalent to

$$(a+c)(b+d) + b^2 + d^2 \geq a^2 + c^2 + 2ac + 2bd$$

$$(a+c)(b+d) - (a+c)^2 + (b-d)^2 \geq 0 \Leftrightarrow$$

$$(a+c)(b+d-a-c) + (b-d)^2 \geq 0 \qquad (7)$$

If $a + c \geq b + d$ then the inequality (6) is true, else the inequality (7) is true. It follows that the inequality (5) is true.

Equality holds if and only if $e = f, a = c, b = d$, hence if and only if $ABCD$ is a square.