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$$E_1=1-\sum_{cyc}\frac{x}{2x+z},\quad E_2=1-\sum_{cyc}\frac{x}{x+2y},x,y,z>0$$

Make a choice: A. $E_1E_2\ge 0$ **B.** $E_1E_2\le 0$

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We will prove that

$$\frac{x}{2x+z}+\frac{y}{2y+x}+\frac{z}{2z+y}\le 1\qquad (1)$$

Since we want to use Bergström inequality, we change " $\le$ " sign to " $\ge$ " sign. The inequality (1) is equivalent to

$$\frac{1}{2}-\frac{x}{2x+z}+\frac{1}{2}-\frac{y}{2y+x}+\frac{1}{2}-\frac{z}{2z+y}\ge\frac{3}{2}-1$$

$$\frac{z}{2x+z}+\frac{x}{2y+x}+\frac{y}{2z+y}\ge 1\qquad (2)$$

Applying Bergström inequality we obtain

$$\frac{z}{2x+z}+\frac{x}{2y+x}+\frac{y}{2z+y}=\frac{z^2}{2zx+z^2}+\frac{x^2}{2xy+x^2}+\frac{y^2}{2yz+y^2}$$
$$\ge\frac{(x+y+z)^2}{x^2+y^2+z^2+2(xy+yz+zx)}=\frac{(x+y+z)^2}{(x+y+z)^2}=1\Rightarrow E_1$$
$$=1-\sum_{cyc}\frac{x}{2x+z}\ge 0.$$

Equality holds if and only if $x=y=z$.

Applying Bergström we obtain

$$\frac{x}{x+2y}+\frac{y}{y+2z}+\frac{z}{z+2x}=\frac{x^2}{x^2+2xy}+\frac{y^2}{y^2+2yz}+\frac{z^2}{z^2+2zx}\ge$$
$$\ge\frac{(x+y+z)^2}{x^2+y^2+z^2+2(xy+yz+zx)}=\frac{(x+y+z)^2}{(x+y+z)^2}=1\Rightarrow E_2=1-\sum_{cyc}\frac{x}{2x+y}\le 0.$$

Equality holds if and only if $x=y=z$.

It follows that $E_1E_2\le 0$.