ROMANIAN MATHEMATICAL MAGAZINE

S.2395 If $a, b, c > 0$, then:

$$\left(\frac{a+b+c}{3\sqrt[3]{abc}}+\frac{3\sqrt[9]{abc}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}\right)\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3\sqrt[9]{abc}}+\frac{3\sqrt[27]{abc}}{\sqrt[9]{a}+\sqrt[9]{b}+\sqrt[9]{c}}\right)\geq 4$$

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We will prove that if $x, y, z > 0$ then

$$\frac{x^9+y^9+z^9}{3x^3y^3z^3}+\frac{3xyz}{x^3+y^3+z^3}\geq 2 \quad (1)$$

which is equivalent to

$$\frac{x^9+y^9+z^9}{3x^3y^3z^3}-1\geq 1-\frac{3xyz}{x^3+y^3+z^3}$$

$$\frac{(x^3+y^3+z^3)((x^3-y^3)^2+(y^3-z^3)^2+(z^3-x^3)^2)}{3x^3y^3z^3}\geq$$

$$\geq\frac{(x+y+z)((x-y)^2+(y-z)^2+(z-x)^2)}{x^3+y^3+z^3}.$$

It suffices to prove that

$$\frac{(x^3+y^3+z^3)(x^2+xy+y^2)^2(x^3-y^3)^2}{3x^3y^3z^3}\geq\frac{(x+y+z)(x-y)^2}{x^3+y^3+z^3}.$$

If $x=y$ then we have equality else it remains to show that

$$(x^3+y^3+z^3)^2(x^2+xy+y^2)^2\geq 3x^3y^3z^3(x+y+z),$$

or

$$(2(x^3+y^3)z^3+z^6)(x^2+xy+y^2)^2>3x^3y^3z^3(x+y+z)$$

$$\Leftrightarrow (2x^3+3y^3+z^3)(x^2+xy+y^2)^2>3x^3y^3(x+y+z).$$

After some calculations, it remains to prove that $3x^4y^3-3x^3y^3z+xy^3z^3\geq 0$, which folows by $AM-GM$ inequality: $x^4y^3+x^4y^3+xy^3z^3\geq 3x^3y^3z$.

Equality holds if and only if $x=y=z$.

For $x=\sqrt[9]{a}, y=\sqrt[9]{b}, z=\sqrt[9]{c}$ in (1):

$$\frac{a+b+c}{3\sqrt[3]{abc}}+\frac{3\sqrt[9]{abc}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}\geq 2 \quad (2)$$

For $x=\sqrt[27]{a}, y=\sqrt[27]{b}, z=\sqrt[27]{c}$ in (1):

$$\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3\sqrt[9]{abc}}+\frac{3\sqrt[27]{abc}}{\sqrt[9]{a}+\sqrt[9]{b}+\sqrt[9]{c}}\geq 2 \quad (3)$$

By multypling $(2),(3)$:

$$\left(\frac{a+b+c}{3\sqrt[3]{abc}}+\frac{3\sqrt[9]{abc}}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}\right)\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3\sqrt[9]{abc}}+\frac{3\sqrt[27]{abc}}{\sqrt[9]{a}+\sqrt[9]{b}+\sqrt[9]{c}}\right)\geq 4$$

Equality if and only if $a=b=c$.