ROMANIAN MATHEMATICAL MAGAZINE

S.2400 If I – incenter of triangle ABC then:

$$IA^4+IB^4+IC^4\ge\frac{(a^2+b^2+c^2)^2}{27}$$

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Since $w_a=\frac{2bc\cos\frac{A}{2}}{b+c}$, by Van Aubel theorem it results that $IA=\frac{bc\cos\frac{A}{2}}{s}$. It follows that

$$IA^2=\frac{b^2c^2}{s^2}\cdot\frac{s(s-a)}{bc}=\frac{bc(s-a)}{s}.$$

We will use Ravi substitutions: $a=y+z,b=z+x,c=x+y,x,y,z>0$.

Since $s=x+y+z$, yields

$$IA^2=\frac{x(z+x)(x+y)}{x+y+z},a^2+b^2+c^2=2(x^2+y^2+z^2+xy+yz+zx).$$

We have to prove that

$$27(x^2(z+x)^2(x+y)^2+y^2(x+y)^2(y+z)^2+z^2(y+z)^2(z+x)^2)^2\ge 4(x^2+y^2+z^2+xy+yz+zx)^2(x+y+z)^2$$

Using the known inequality $3(m^2+n^2+p^2)\ge(m+n+p)^2$, it is enough to prove that

$$9\big(x(z+x)(x+y)+y(x+y)(y+z)+z(y+z)(z+x)\big)^2\ge 4(x^2+y^2+z^2+xy+yz+zx)^2(x+y+z)^2$$

which is equivalent to

$$3\big(x(z+x)(x+y)+y(x+y)(y+z)+z(y+z)(z+x)\big)\ge 2(x^2+y^2+z^2+xy+yz+zx)(x+y+z)$$

$$3(x^3+y^3+z^3)+3(x+y+z)(xy+yz+zx)\ge 2(x^2+y^2+z^2)(x+y+z)+2(x+y+z)(xy+yz+zx)$$

$$3(x^3+y^3+z^3)+(x+y+z)(xy+yz+zx)\ge 2(x^2+y^2+z^2)(x+y+z)$$

$$x^3+y^3+z^3+3xyz\ge x^2y+xy^2+y^2z+yz^2+z^2x+zx^2$$

$$x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\ge 0\quad (1)$$

The inequality (1) is Schur's inequality.

Equality holds if and only if $x=y=z$, and the triangle ABC is equilateral.