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S.2401 If $m, n > 0$ the in ΔABC the following relationship holds:

$$4\sqrt{3}(m+n)\cdot\frac{r}{R} \le \sum_{cyc}\frac{ma+nb}{r_c} \le \frac{3R(m+n)}{F}\cdot\sqrt{9R^2-s^2}$$

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Since $r_a = \frac{F}{s-a}$, we obtain:

$$\sum_{cyc}\frac{ma+nb}{r_c} = \frac{ma(s-c)+nb(s-c)+mb(s-a)+nc(s-a)+mc(s-b)+na(s-b)}{F} =$$

$$= \frac{m\big(s(a+b+c)-(ab+bc+ca)\big)+n\big(s(a+b+c)-(ab+bc+ca)\big)}{F} =$$

$$= \frac{(m+n)((a+b+c)^2-2(ab+bc+ca))}{2F} = \frac{(m+n)(a^2+b^2+c^2)}{2F}.$$

It results that we have to prove that

$$4\sqrt{3}\cdot\frac{r}{R} \le \frac{a^2+b^2+c^2}{2F} \le \frac{3R}{F}\cdot\sqrt{9R^2-s^2} \quad (1)$$

By Euler inequality $R \ge 2r$ and Ionescu-Weitzenbock's inequality $a^2+b^2+c^2 \ge 4\sqrt{3}\cdot F$,

it follows that:

$$\frac{a^2+b^2+c^2}{2F} \ge 2\sqrt{3} \ge 4\sqrt{3}\cdot\frac{r}{R}.$$

The right inequality in (1) is equivalent to

$$a^2+b^2+c^2 \le 6R\sqrt{9R^2-s^2} \Leftrightarrow \left(a^2+b^2+c^2\right)^2 \le 36R^2(9R^2-s^2)$$

Using the known inequality $a^2+b^2+c^2 \le 9R^2$, it suffices to prove that

$$81R^4 \le 36R^2\left(9R^2-s^2\right) \Leftrightarrow 9R^2 \le 36R^2-4s^2 \Leftrightarrow a+b+c \le 3\sqrt{3}R,$$

which is item 5.5 from $[1]$.

Equality for both sides holds if and only if ΔABC is equilateral.

[1] O. Bottema, *Geometric Inequalities*, Groningen 1969